# Efficient c-planarity testing algebraically

Radoslav Fulek[1⋆], Jan Kynčl[1⋆⋆], Igor Malinović[2], and Dömötör Pálvölgyi[3⋆⋆⋆]

[1] Charles University, Prague, Czech Republic,
radoslav.fulek@gmail.com, kyncl@kam.mff.cuni.cz
[2] EPFL Lausanne,
igor.malinovic@epfl.ch
[3] Institute of Mathematics, Eötvös University, Budapest,
domotorp@gmail.com

**Abstract.** We generalize the strong Hanani-Tutte theorem to clustered graphs with two disjoint clusters, and show that an extension of our result to flat clustered graphs with three disjoint clusters is not possible. We also give a new and short proof for a result by Di Battista and Frati about efficient c-planarity testing of an embedded flat clustered graph with small faces based on the matroid intersection algorithm.

**Keywords:** Hanani-Tutte theorem, Matroid intersection theorem, planarity testing

## 1 Introduction

Nowadays, it is a folklore result that testing whether a graph admits an edge crossing free drawing in the plane can be done in polynomial time. However, for various notions of planarity the existence of an efficient testing algorithm was neither proved nor disproved assuming that $\mathsf{P} \neq \mathsf{NP}$. One of the most prominent of such planarity notions is *clustered planarity*. Roughly speaking, an instance of the problem in this case is a graph whose vertices are partitioned into clusters. A *clustered graph* is *clustered planar* (or briefly *c-planar*) if it can be drawn in the plane so that the vertices from the same cluster belong to the same region and no edge cuts through a region that a cluster corresponds to.

More precisely, a *clustered graph* is a pair $(G, T)$ where $G = (V, E)$ is a graph and $T$ is a rooted tree whose set of leaves is the set of vertices of $G$. The non-leaf vertices of $T$ correspond to the clusters. For $v \in V(T)$, let $T_v$ denote the subtree of $T$ rooted at $v$. A *drawing* of $G$ is a representation of $G$ in the plane such that every vertex is represented by a unique point and every edge $e = uv$ is a Jordan arc joining two points that represents $u$ and $v$. We assume that in a drawing no edge passes through a vertex, no two edges touch and every pair of edges cross in finitely many points. The drawing of

⋆ The author gratefully acknowledge support from the Swiss National Science Foundation Grant No. 200021-125287/1 and ESF Eurogiga project GraDR as GAČR GIG/11/E023.

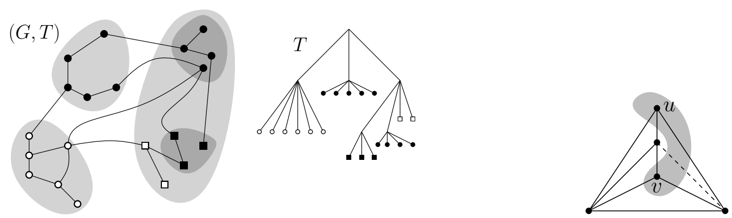
⋆⋆ The author gratefully acknowledge support from ESF Eurogiga project GraDR as GAČR GIG/11/E023.

⋆⋆⋆ Supported by Hungarian National Science Fund (OTKA), under grant PD 104386 and under grant NN 102029 (EUROGIGA project GraDR 10-EuroGIGA-OP-003) and the János Bolyai Research Scholarship of the Hungarian Academy of Sciences.

a graph is an *embedding* if no two edges of the graph cross. The clustered graph $(G, T)$ is *c-planar* if $G$ can be embedded in the plane so that

(i) for every $v \in V(T)$, there is a topological disc $d(T_v)$ containing all the leaves of $T_v$ and no other vertices of $G$,
(ii) if $u \in T_v$, then $d(T_u) \subseteq d(T_v)$,
(iii) if $u_1$ and $u_2$ are children of $v$ in $T$, then $d(T_{u_1})$ and $d(T_{u_2})$ are internally disjoint, and
(iv) for every $v \in V(T)$, every edge of $G$ intersects the boundary of the disc $d(T_v)$ at most once.

A *clustered drawing (resp. embedding)* of a clustered graph $(G, T)$ is a drawing of $(G, T)$ (resp. embedding) satisfying (i)-(iv). See Fig. 1(a) and Fig. 1(b) for an illustration.




(a) (b)

**Fig. 1.** (a) A clustered embedding of a clustered graph $(G, T)$, and its tree $T$; (b) A clustered graph $(G, T)$, $G = K_5 - e$, with two clusters, where one cluster contains $u$ and $v$, that is not c-planar.

The notion of clustered planarity appeared for the first time in the literature in the work of Feng, Cohen and Eades [10, 11] under the name of c-planarity. Since then an efficient algorithm for c-planarity testing or embedding has been discovered only in some special cases. The general problem whether the c-planarity of a clustered graph can be tested in polynomial time is wide open.

*Solved cases* A clustered graph $(G, T)$ is *c-connected* if every cluster of $(G, T)$ induces a connected subgraph. In order to test a c-connected clustered graph $(G, T)$ for c-planarity, it is enough to test whether there exists an embedding of $G$ in which for every $v \in V(T)$ all vertices $u \in V(G)$ such that $u \notin V(T_v)$ are drawn in a single face of the subgraph of $G$ corresponding to $T_v$ [11]. This condition can be tested in polynomial time and remains true if the clustered graph consists only of two non-trivial clusters forming a partition of the vertex set regardless of the cluster connectivity. Cortese et al. [4] gave a structural characterization of c-planarity for c-connected clustered graphs and provided a linear-time algorithm. Gutwenger et al. devised an efficient algorithm in a more general case of *almost connected* clustered graphs [13]. Biedl [2] gave a linear-time algorithm for several variants of c-planarity with two clusters, including the case of straight-line or $y$-monotone drawing. Other special cases of c-planarity testing were treated in [5, 15]. See [6] for further references.

*Hanani-Tutte theorem* Hanani-Tutte theorem [14, 25] is a classical result that provides an algebraic characterization of planarity with interesting algorithmic consequences. The theorem says that a graph is planar as soon as it can be drawn in the plane so that no pair of independent edges crosses an odd number of times. Moreover, its weaker variant, also known as the weak Hanani-Tutte theorem [3, 19, 21], states that if we have a drawing of a graph $G$ so that every pair of edges cross an even number of times then $G$ has an embedding that preserves the cyclical order of edges at vertices from the given drawing. Thus, the weak variant is slightly stronger in some sense. On the other hand, the strong Hanani-Tutte theorem was used to prove the correctness of a planarity testing algorithm (see e.g [[23], Section 1.4.2])

The weak version seems to have direct algorithmic consequences in planarity testing only for sub-cubic graphs.

Other variants of the Hanani-Tutte theorem were proved for surfaces of higher genus [20, 22], $x$-monotone drawing [12], partially embedded planar graphs, and simultaneously embedded planar graphs [24]. Here, it is interesting to note that we are not aware of any variant of Hanani-Tutte theorem, for which it is proved that either the weak or the strong version holds. Usually it was either proved that both variants holds, e.g. [12, 20], or as in the case of general closed surfaces, only the weak version was proved [22], and the general version is open. See [23] for a recent survey on its applications and related results.

We prove a variant of Hanani-Tutte theorem for cluster graphs consisting only of two clusters. Our results give polynomial-time algorithms for c-planarity testing in the corresponding special cases based on linear algebra. The algorithms are analogous to the planarity testing algorithm based on the Hanani-Tutte theorem. The running time of our algorithm for the case of two clusters can be only proved to be in $O(|V(G)|^6)$ (by the analysis from [[23], Section 1.4.2]), which does not beat the linear time algorithm from [2]. Thus, the algorithm has the same running as a straightforward Hanani-Tutte based algorithm for planarity testing. Nevertheless, we think that our algorithm is much simpler and the proof of its correctness less cumbersome. We remark that there exist more efficient algorithms for planarity testing based on the Hanani-Tutte theorem such as the one in [8], which runs in a linear time. Hence, Hanani-Tutte based approach proved to have also some practical aspects. A slightly different Hanani-Tutte based approach towards clustered planarity was taken recently in [24].

*Notation* In the present paper we assume that $G = (V, E)$ is a (multi)graph. We use a shorthand notation $G-v$, and $G\cup E'$, respectively, for $(V-\{v\}, E-\{e \in E|\ e = vw\})$, and $(V, E \cup E')$. If it leads to no confusion we do not distinguish between a vertex or an edge and its representation in the drawing and we will use the words "vertex" and "edge" in both contexts. Similarly, we will be using the word "cluster" for both the topological discs and a finite set of vertices. The graph is *planar* if it admits an embedding. The *rotation* at a vertex $v$ is the clockwise cyclic order of the end pieces of edges incident to $v$. The *rotation system* of a graph is the set of rotations at all its vertices. We say that two embeddings of a graph are the *same* if they have the same rotation system up to switching the orientations of all the rotations simultaneously. We say that a pair of edges in a graph is *independent* if they do not share a vertex. We say that a pair of edges in a drawing of a graph is *even* if they cross an even number of times. A drawing of a graph is *even* if every pair of edges in the drawing cross an even number of times.

*Hanani-Tutte for clustered graphs* A clustered graph $(G, T)$ is two-clustered if the root of $T$ contains exactly two siblings having only leaves as children. We show the following generalization of the weak Hanani-Tutte theorem for graphs whose vertex set is partitioned into two sets, $A$ and $B$.

**Theorem 1.** *If a two-clustered graph $(G, T)$ admits an even clustered drawing then $(G, T)$ is c-planar. Moreover, there exists a clustered embedding of $(G, T)$ with the same rotation system as in the even clustered drawing.*

Analogously we extend the strong version of the Hanani-Tutte theorem. A drawing of a graph is *independently even* if every pair of independent edges in the drawing cross an even number of times.

**Theorem 2.** *If a two-clustered graph $(G, T)$ admits an independent even clustered drawing then $(G, T)$ is c-planar.*

We say that a cluster graph $(G, T)$ is *flat* if no non-root cluster of $(G, T)$ has a non-trivial sub-cluster; that is, every root-leaf path in $T$ has exactly three vertices, or every root-leaf path in $T$ has exactly two vertices, in which case the graph is not even clustered.

On the other hand, in Section 5 we give an example of a flat clustered cycle on three clusters that is not c-planar, but admits a clustered drawing in which every pair of edges crosses an even number of times. Thus, a straightforward extension of Theorem 1 or Theorem 2 to more than two clusters is not possible.

*Embedded clustered graph* A pair $(\mathcal{D}(G), T)$ is an *embedded clustered graph* if $(G, T)$ is a clustered graph and $\mathcal{D}(G)$ is an embedding of $G$ in the plane. The embedded clustered graph $(\mathcal{D}(G), T)$ is *c-planar* if there exists a clustered embedding of $(G, T)$ in which the embedding of $G$ is the same as $\mathcal{D}(G)$. In the rest of the present paper we will use a shorthand notation $(G, T)$ for an embedded clustered graph instead of $(\mathcal{D}(G), T)$.

We give an alternative polynomial time algorithm for deciding c-planarity in the case of embedded flat clustered graphs with small faces by reproving a result of Di Battista and Frati [1]. Our algorithm does not outperform the one from [1] in the running time, since that one has a linear running time in $|V(G)|$, while the running time of ours is $O(|V(G)|^{3.5})$ by [7]. However, our algorithm is based on the Matroid intersection theorem, and thus, the algorithm itself and the proof of its correctness are much simpler.

**Theorem 3.** *If $(G', T')$ is an embedded flat clustered graph such that all its faces are incident to at most five vertices, then we can decide whether $(G', T')$ has a c-planar drawing with the same embedding in polynomial time.*

*Organization* The rest of the paper is organized as follows. In Section 2 we describe an algorithm for c-planarity testing of two-clustered graphs based on Theorem 2. In Section 3, we give the proof of Theorem 1. In Section 4, we give the proof of Theorem 2. In Section 5 we provide a counter-example to a variant of the Hanani-Tutte theorem for clustered cycles on three clusters. In Section 6 we prove Theorem 3.

We conclude with some remarks in Section 7.

## 2 Algorithm

Let $(G, T)$ be a clustered graph belonging to a class of clustered graphs for which the strong Hanani-Tutte theorem holds. As we proved the strong variants of Hanani-Tutte theorem for two-clustered graphs, $(G, T)$ is assumed to be two-clustered.

Our algorithm is an adaption of the algorithm for planarity testing from [[23], Section 1.4.2]. Hence, the algorithm tests whether we can continuously deform a given clustered drawing $\mathcal{D}$ of $(G, T)$ into an independently even clustered drawing $\mathcal{D}'$ of $(G, T)$. By the corresponding variant of the strong Hanani-Tutte theorem the algorithm is correct.

Observe that during our deformation the parity of crossings between a pair of edges is affected only when an edge $e$ passes over a vertex $v$, in which case we change the parity of crossings of $e$ with all the edges adjacent to $v$. Let us call such an event an *edge-vertex switch*. Thus, for our purpose the continuous deformation of $\mathcal{D}$ can be represented by a set $S$ of edge-vertex switches. In $S$, an edge-vertex switch of an edge $e$ with a vertex $v$ is represented as the ordered pair $(e, v)$. Let $S(e)$ denote the set of vertices $v$ of $V$ such that $(e, v) \in S$. Similarly, a clustered drawing of $(G, T)$ can be represented as a vector **v** in the vector field $\mathbb{Z}_2^M$, where $M$ denotes the number of unordered pairs of independent edges, i.e., edges not sharing a vertex, such that the component of **v** corresponding to a pair $e$ and $f$ is 1 iff $e$ and $f$ cross an odd number of times. An edge-vertex $(e, v)$ switch is represented as a vector **w** in the vector field $\mathbb{Z}_2^M$ so that the component of **v** corresponding to a pair $e$ and $f$ is 1 iff $f$ is incident to $v$.

The algorithm tests if we can obtain the all-zero vector from **v** by adding to it some vectors corresponding to edge-vertex switches. However, we have to be a bit careful here. Since the resulting drawing $\mathcal{D}'$ of $(G, T)$ has to be a clustered drawing, for an edge $e$ we allow to perform edge-vertex switches freely only with vertices belonging exactly to the same cluster(s) as the endpoints of $e$. For the vertices belonging to the other cluster, $e$ performs the edge-vertex switches either with all of the vertices belonging to a particular cluster, or with none of them.

The previous paragraph captures all the differences between the planarity testing and $c$-planarity testing algorithm based on the strong Hanani-Tutte theorem. It is easy to see that the algorithm amounts to solving a system of linear equations over $\mathbb{Z}_2$ of polynomial size.

## 3 Weak Version

First, we prove a stronger version of a special case of Theorem 1 in which $G$ is a bipartite graph with the parts corresponding to clusters. In this stronger version, which is an easy consequence on the weak Hanani-Tutte theorem, we assume only the existence of an arbitrary even drawing of $G$ that does not have to be a clustered drawing.

Let $C$ denote a closed Jordan curve drawn in the plane. We say that a pair of points $p$ and $q$ is *separated* by $C$ if $p$ and $q$ belong to distinct connected components of the complement of $C$ in the plane.

**Lemma 1.** *Let $(G, T)$ denote a two-clustered bipartite graph in which clusters induce independent sets. If $G$ admits an even drawing then $(G, T)$ is c-planar. Moreover, there* *exists a clustered embedding of* $(G, T)$ *with the same rotation system as in the given even clustered drawing of* $G$.

**Proof of Theorem 1**

The proof is inspired by the proof of the weak Hanani-Tutte theorem from [21].

Let $A$ and $B$ denote the partition of $V$ corresponding to the clusters of $(G, T)$. We assume that $G$ is connected, since we can embed the connected parts separately. We proceed by induction on the number of vertices. Thus, suppose that we have an even clustered drawing of $(G, T)$.

First, we discuss the inductive step. If we have an edge $e$ between two vertices in the same part (either $A$ or $B$), we contract $e = uv$ by moving $v$ along $e$ towards $u$ in the initial drawing of $(G, T)$ while dragging all the other edges incident to $v$ along $e$ as well. By the fact that our drawing is even, this operation does not introduce any pair of edges crossing an odd number of times, and it also does not change the rotation at any vertex. Thus, the resulting drawing is still a clustered drawing. We might create a self crossing of an edge, though. However, a self-crossing of an edge can be easily eliminated by cutting the edge at the self-crossing and reconnecting the severed ends in an appropriate way. Finally, we apply the induction hypothesis and decontract the contracted edge, which can be done without introducing any edge crossing.

In the base step, $G$ is a (multi)graph consisting of a bipartite graph $H$ with parts $A$ and $B$ and possible loops at some vertices. By Lemma 1, we can embed $H$. It remains to embed the loops. Note that after the contractions, no loop crosses a boundary of a cluster. Each loop $l$ divides the rotation at its corresponding vertex $v(l)$ into two intervals. We claim that one of these intervals contains no end piece of an edge connecting $A$ with $B$. Indeed, otherwise $l$ would cross some edge of $H$ an odd number of times. Call such an interval a *good* cyclic interval in the rotation at $v(l)$. Observe that there are no two loops $l_1$ and $l_2$ with $v(l_1) = v(l_2) = v$ whose end-pieces would have the order $l_1, l_2, l_1, l_2$ in the rotation at $v$, as otherwise the two loops would cross an odd number of times. Hence, at each vertex the good intervals of every pair of loops are either nested or disjoint.

We use induction on the number of loops to draw all the loops at a given vertex $v$ without crossings and without changing the rotation at $v$. For the induction step, we remove a loop $l$ whose good cyclic interval in the rotation at $v$ is inclusion minimal. Such an interval contains only the two end-pieces of $l$. By induction hypothesis, we can embed the rest of the loops without changing the rotation at $v$. Finally, we can draw $l$ in a close neighborhood of $v$ within the face corresponding to the good interval of $l$. This concludes our discussion of the base step of the induction and the proof of the theorem.

## 4 Strong Version

Let $(G, T)$ denote a two-clustered graph. Let $A$ and $B$ denote the partition of $V = V(G)$ corresponding to the clusters of $(G, T)$. Let $G[V']$, $V' \subseteq V$, denote the subgraph of $G$ induced by $V'$. The hypothesis of Theorem 2 allows us to apply the strong Hanani-Tutte theorem on the graph $G$ thereby getting an embedding of $G$. The problem is that in such an embedding $G[B]$ does not have to be contained in a single face of $G[A]$ and vice-versa. Hence, we cannot guarantee that a clustered embedding of $(G,T)$ exists so easily.

We first present a lemma which is an easy consequence of the strong Hanani-Tutte theorem. Later in the proof of Theorem 2 the lemma allows us to concentrate only on graphs $G$ in which both $G[A]$ and $G[B]$ do not induce a cycle with a subdivided chord.

**Lemma 2.** *If $(G,T)$ admits an independent even clustered drawing then every connected component $C$ of $G[A]$ (resp. $G[B]$) admits an embedding such that all the vertices $v \in V(C)$, that are incident to an edge having the other end vertex in $B$ (resp. $A$), are incident to the single face.*

*Proof.* Let $C'$ denote the union of $C$ of $G[A]$ with all the edges connecting it with $B$. By contracting all the vertices of $C'$ belonging to $B$ to a single vertex $v$ we obtain a graph, which we denote by $C''$. Observe that we can still use the strong Hanani-Tutte theorem on $C''$ and thereby we get an embedding $\mathcal{D}$ of $C''$ inducing the desired embedding of $C$. Indeed, we can take the independent even drawing of $C''-v$, and draw all the edges in $C''$ incident to $v$ so that inside the cluster corresponding to $A$ they closely follow the former edges between $A$ and $B$ they correspond to. Hence, we do not introduce any pair of independent edges crossing an odd number of times.

### 4.1 Proof of Theorem 2

The proof is inspired by the proof of the strong Hanani-Tutte theorem from [21] and its outline is as follows.

Let $G'$ denote a subgraph of $G$ such that both $G[A]$ and $G[B]$ do not contain a cycle with a subdivided chord. We apply the strong Hanani-Tutte theorem on a graph which is constructed from $G'$ by turning certain cycles in $G[A]$ and $G[B]$ into wheels, and by splitting certain vertices of $G'$ into edges. The wheels in $G'$ serve double purpose. First, they guarantee that everything that was removed from $G$ in order to obtain $G'$ could be inserted back. Second, they allow us to embed $G'$ such that $G[A]$ is drawn in the outer-face of $G[B]$ and vice-versa.

Let $C_1,\ldots,C_k$ denote the connected components of $G[A]$ and $G[B]$. By Lemma 2 we find an embedding $\mathcal{D}(C_i)$ of each connected component $C_i$ of $G[A]$ (resp. $G[B]$) such that all the vertices $v \in V(C_i)$, that are incident to an edge having the other end vertex in $B$ (resp. $A$), are incident to the single face, let's say, the outer-face. Let $C_i'$ denote the subgraph of $C_i$ we get by deleting from $C_i$ all the vertices and edges not incident to the outer-face of $C_i$ in $\mathcal{D}(C_i)$.

Let $G'$ denote the subgraph of $G$ that is the union of $C_i'$-s from the previous paragraph. Let $\mathcal{D}'$ denote the drawing of $G'$ that we obtain from the drawing of $G$, whose existence is guaranteed by the hypothesis of the theorem, by deleting the edges and vertices of $G$ not belonging to $G'$. Thus, every pair of non-adjacent edges in $\mathcal{D}'$ cross an even number of times.

In what follows we process the cycles of $G'[A]$ and $G'[B]$ one by one. We will be modifying $G'$ and therefore also the drawing $\mathcal{D}'$. At each stage of this process some cycles of $G'[A]$ and $G'[B]$ will be labeled as processed and some will be labeled as unprocessed. We start with all the cycles in $G'[A]$ and $G'[B]$ being labeled as unprocessed.

Let $C$ denote an unprocessed cycle in $G'[A]$. First, suppose that $C$ does not share a vertex with an already processed cycle. The following argument also applies if $C$ is in $G'[B]$.

Let us two-color the connected regions in the complement of $C$ so that no two regions sharing a non-trivial part of the boundary receive opposite colors. We say that a point not lying on $C$ is "outside" of $C$ if it is contained in the region in the complement of $C$ having the same color as the unbounded region. Otherwise, such a point is "inside" of $C$. We correct the rotations at the vertices of $C$ in the drawing $\mathcal{D}'$ so that all the edges of $C$ cross every other edge an even number of times. Note that every path joining $C$ with a vertex in $B$ and otherwise vertex disjoint from $C$ has to start at a vertex of $C$ on the "outside" of $C$. We add a vertex $v_C$ into $A$ that is drawn very close to an arbitrary vertex of $C$ "inside" of $C$. We connect $v_C$ with all the vertices of $C$ by the edges that closely follow edges of $C$ on the "inside". Note that the new edges connecting $v_C$ can introduce an odd crossing pair only with an edge $e$ that starts at a vertex $v$ of $C$ on the "inside" of $C$.

Since $G'[A]$ does not contain a cycle with a subdivided chord, it follows that the vertex $v$ is a cut vertex in $G'[A]$ and that the endpoint of $e$ different from $v$ belongs to a connected component $C'$ of $G'[A - v]$ none of whose vertices is joined with a vertex in $B$ by an edge in $G'$. Thus, we can shrink the drawing of $G'[V(C') \cup v]$ so that $G'[V(C') \cup v]$ is drawn very close to $v$ and none of its edges crosses an edge in the rest of the graph. Moreover, by shrinking $G'[V(C') \cup v]$ we do not introduce a pair of edges crossing an odd number of times. We label all the cycles in $G'[V(C') \cup v]$ as processed. By repeating this for all the troublesome cut-vertices of $C$ we modify $\mathcal{D}'$ so that none of the edges incident to $v_C$ crosses another edge an odd number of times, and afterwards we label $C$ as processed.

If $C$ shares a vertex $v$ with an already processed cycle $C_p$ we split the vertex $v$ as follows. We replace the vertex $v$ with two new vertices $v'$ and $v''$ belonging to $A$ such that $v'$ is connected by an edge with $v''$ and with the neighbors of $v$ starting on the "inside" of $C_p$ and on $C_p$, and $v''$ is connected to the rest of the neighbors of $v$. The cycle that was obtained from $C_p$ by replacing $v$ with $v'$ is then labeled as processed. Note that we can do such a vertex-splitting in $\mathcal{D}'$ without introducing any pair of edges crossing an odd number of times by drawing $v'$ and $v''$ very close to the spot where $v$ was drawn.

Since $G'[A]$ does not contain a cycle with a subdivided chord, after performing a finite number of vertex-splits all the cycles in $G'[A]$ become vertex disjoint. Thus, if we still have an unprocessed cycle in $G'[A]$, we eventually obtain a modified graph, in which there exists an unprocessed cycle not sharing a vertex with an already processed cycle.

Let $G''$ denote the graph we obtain from $G'$ after processing all the cycles of $G'[A]$ and $G'[B]$. By applying the strong Hanani-Tutte theorem on $G''$ we obtain an embedding which can be, due to the wheels, easily modified so that $G''[A]$ is drawn in the outer-face of $G''[B]$ and vice-versa. In the resulting embedding we delete all the vertices $v_C$ that were added for the cycles $C$ in $G'[A]$ and $G'[B]$, and we contract the edges between the pairs of vertices that were obtained by vertex-splits. We perform contractions by dragging edges incident to a vertex of a contracted edge $e$ along $e$ without changing the rotations at the vertex obtained by the contraction. Thus, we obtain an embedding of $G'$

in which $G'[A]$ is drawn in the outer-face of $G'[B]$ and vice-versa, as well. By inserting the removed parts of $G$ back to $G'$ we obtain an embedding of $G$ in which $G[A]$ is drawn in the outer-face of $G[B]$ and vice-versa. Note that this can be done without introducing any edge crossing. Thus, by the observation from [11] discussed in the introduction the theorem follows.

## 5 Counterexample on Three Clusters

In this section we construct a family of even clustered drawing of flat clustered cycles on three clusters that are not clustered planar. Thus, a straightforward variant of Hanani-Tutte on three clusters is not possible. In our clustered drawing the clusters are drawn as regions bounded by a pair of rays emanating from the same vertex $p$. We call $p$ the *center* of our drawing.

Our family of counter-examples are clustered cycles winding $k$ times, where $k$ is odd, around the center of the drawing.

In Fig. 2, we give a counter-example for $k = 5$, which can be easily generalized for any odd number $k > 1$.

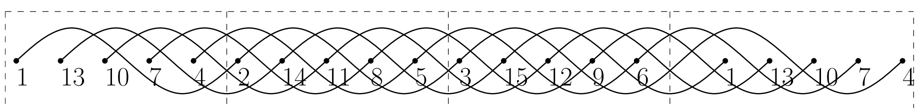



**Fig. 2.** A counter-example to the variant of the Hanani-Tutte theorem for three clusters. The underlying graph is a cycle on 15 vertices. The vertices are labeled by natural numbers in correspondence with their appearance on the cycle. Due to a better readability the clusters are separated by vertical lines and the leftmost and the rightmost cluster need to be identify in the actual drawing.

## 6 Small faces

In this section we show that c-planarity can be decided in polynomial time for embedded flat clustered graphs if all faces are incident to at most five vertices. This reproves a result of Di Battista and Frati [1]. Our approach seems quite different from theirs, as we use (a corollary of) the matroid intersection theorem [9, 16], that the largest common independent set of two matroids can be found in polynomial time. See e.g. [18] for further references.

Let $(G, T)$ denote a flat embedded clustered graph. A *saturator* of $(G, T)$ is a subset $F$ of $\binom{V}{2}$ disjoint from $E(G)$ such that $(G \cup F, T)$ is planar, every cluster of $(G \cup F, T)$ is connected, and the edges in $F$ can be embedded so that every cluster of $(G \cup F, T)$ is in the outer-face of every other cluster. We have the following simple observation regarding saturators.

**Observation** *A flat embedded clustered graph* $(G, T)$ *is c-planar if and only if* $(G, T)$ *has a saturator.*

From Observation we can easily conclude the following lemma.

**Lemma 3.** *An embedded flat clustered graph* $(G, T)$*, all of whose faces are incident to at most five vertices, can be augmented by adding edges into an embedded flat clustered graph* $(G', T')$ *such that* $(G, T)$ *is c-planar if and only if* $(G', T')$ *is c-planar, and the following holds for* $(G', T')$*. If* $(G', T')$ *is c-planar then* $(G', T')$ *has a saturator* $F$ *whose edges can be embedded so that at most one edge of* $F$ *is inside each face of* $G'$*.*

**Proof of Theorem 3**

We give an algorithm for deciding c-planarity for flat embedded clustered graphs satisfying the hypothesis of the claim.

By an algorithmic version of Lemma 3, from the given embedded flat clustered graph $(G', T')$ we obtain a new embedded graph $(G, T)$ such that every minimal saturator of $(G, T)$ has at most one edge inside each face and $(G, T)$ is c-planar if and only if $(G', T')$ is c-planar. This can be done easily in a linear time in the number of vertices, which follows from the proof of Lemma 3 omitted in this abstract. Thus, it is enough to show that we can decide c-planarity of $(G, T)$ in polynomial time.

Since $(G, T)$ is a flat clustered graph, by Observation, it is enough decide whether we can saturate $G$ so that all the clusters are connected, and every cluster is drawn in the outer-face of every other cluster. Clearly, the latter can be tested in a linear time in the number of vertices. In order to test the existence of a saturator let us define the two matroids for which we will use the matroid intersection algorithm. The ground set of each matroid is the set of non-edges of $G$, denoted by $\overline{E}$.

The first matroid, $M_1$, is the direct sum of graphic matroids constructed for each cluster. More precisely, denote the clusters by $C_i$, $i = 1, \ldots, k$, and let $v \sim_i u$ if $u$ and $v$ are connected in $G[C_i]$.

Denote by $G_i$ the multigraph obtained from $\overline{G} = (V, \overline{E})$ by deleting the vertices not in $C_i$, contracting the $\sim_i$-equivalent vertices into new vertices, and deleting all loops. Now, the ground set of the graphic matroid $M(G_i)$ can be identified with the set of edges from $\overline{E}$ that go between two vertices from $C_i$ belonging to distinct connected components of $C_i$. The rank of $M(G_i)$ is the number of vertices of $G_i$ minus one.

Since the matroids $M(G_i)$, $i = 1, \ldots, k$, are pairwise disjoint, their direct sum, $M_1$, is also a matroid and its rank is the sum of the ranks of $M(G_i)$-s. The second matroid, $M_2$, is a partition matroid. A subset of $\overline{E}$ is independent in $M_2$ if it has at most one edge in every face of $G$.

Let $M$ be the intersection of $M_1$ and $M_2$. If $M$ has the same rank as $M_1$ then there exists a saturator of $(G, T)$ that has at most one edge inside each face. Thus, $(G, T)$ is c-planar by Observation, and that in turn implies that $(G', T')$ is c-planar as well. On the other hand, if $(G', T')$, and hence $(G, T)$, is c-planar then there exists a minimal saturator $F$ of $G$ that has at most one edge inside each face by the property of $G$ guaranteed by Lemma 3. Thus, $F$ witnesses the fact that the rank of $M_1$ and the rank of $M$ are the same. Hence, $M$ has the same rank as $M_1$ if and only if $(G, T)$ is c-planar and the theorem follows by the matroid intersection algorithm.

## 7 Concluding remarks

By the construction in Section 5 we cannot hope for the fully general variant of the Hanani-Tutte theorem for clustered graphs. Nevertheless, it is still interesting to ask, whether the weak or strong Hanani-Tutte theorem holds in the case of flat clustered graphs, if the graph obtained by contracting clusters does not contain a cycle (after deleting loops and multiple edges).

More formally, given a flat clustered graph $(G,T)$ let $G'$ denote the simple graph whose vertices correspond to clusters of $(G,T)$, in which two distinct vertices $u$ and $v$ are joined by an edge if and only if there exists an edge in $G$ with one endpoint in the cluster corresponding to $u$ and one endpoint in the cluster corresponding to $v$.

**Conjecture** *If $G'$ does not contain a cycle we have the following. If $(G,T)$ admits an (independent) even clustered drawing then $(G,T)$ is c-planar.*

We note that even a weaker variant of the weak version of Conjecture in which we assume that $G'$ is a path does not seem to be easy.

It is also still possible that an efficient Hanani-Tutte based algorithm for c-planarity testing in the general case can be constructed, which is a view supported in [24].

Note that our proof from Section 6 fails if the graph has hexagonal faces. If in a hexagon $abcdef$ we have $\{b,e\} \subseteq C_1$, $\{a,c\} \subseteq C_2$ and $\{d,f\} \subseteq C_3$, then $M_2$ will no longer be a partition matroid. We wonder if this difficulty can be overcome or rather could lead to NP-hardness.

## References

1. Giuseppe Di Battista and Fabrizio Frati. Efficient c-planarity testing for embedded flat clustered graphs with small faces. In Seok-Hee Hong, Takao Nishizeki, and Wu Quan, editors, *Graph Drawing*, Lecture Notes in Computer Science, pages 291–302. Springer, 2007.
2. Therese C. Biedl. Drawing planar partitions III: Two constrained embedding problems, 1998.
3. Grant Cairns and Yury Nikolayevsky. Bounds for generalized thrackles. *Discrete Comput. Geom.*, 23(2):191–206, 2000.
4. Pier Francesco Cortese, Giuseppe Di Battista, Fabrizio Frati, Maurizio Patrignani, and Maurizio Pizzonia. C-planarity of c-connected clustered graphs. *J. Graph Algorithms Appl.*, 12(2):225–262, 2008.
5. Pier Francesco Cortese, Giuseppe Di Battista, Maurizio Patrignani, and Maurizio Pizzonia. Clustering cycles into cycles of clusters. *J. Graph Algorithms Appl.*, 9(3):391–413, 2005. also in proc. of Graph Drawing 2004.
6. Pier Francesco Cortese and Giuseppe Di Battista. Clustered planarity (invited lecture). In *Twenty-first annual symposium on Computational Geometry (proc. SoCG 05)*, ACM, pages 30–32, 2005.
7. William H. Cunningham. Improved bounds for matroid partition and intersection algorithms. *SIAM Journal on Computing*, 15(4):948–957, 1986.
8. Hubert de Fraysseix and Pierre Rosenstiehl. A characterization of planar graphs by Trémaux orders. *Combinatorica*, 5(2):127–135, 1985.
9. Jack Edmonds. Submodular functions, matroids, and certain polyhedra. In Michael Jünger, Gerhard Reinelt, and Giovanni Rinaldi, editors, *Combinatorial Optimization*, volume 2570 of *Lecture Notes in Computer Science*, pages 11–26. Springer, 2001.

10. Qing-Wen Feng, Robert F. Cohen, and Peter Eades. How to draw a planar clustered graph. In *COCOON*, pages 21–30, 1995.
11. Qing-Wen Feng, Robert F. Cohen, and Peter Eades. Planarity for clustered graphs. In *ESA*, pages 213–226, 1995.
12. Radoslav Fulek, Michael Pelsmajer, Marcus Schaefer, and Daniel Štefankovič. Hanani-Tutte, monotone drawings and level-planarity. *Thirty Essays in Geometric Graph Theory, J. Pach ed.*, pages 263–288, 2012.
13. Carsten Gutwenger, Michael Jünger, Sebastian Leipert, Petra Mutzel, Merijam Percan, and René Weiskircher. Advances in c-planarity testing of clustered graphs. In Michael Goodrich and Stephen Kobourov, editors, *Graph Drawing*, volume 2528 of *Lecture Notes in Computer Science*, pages 220–236. Springer Berlin, Heidelberg, 2002.
14. Chaim Chojnacki (Haim Hanani). Über wesentlich unplättbare Kurven im drei-dimensionalen Raume. *Fundamenta Mathematicae*, 23:135–142, 1934.
15. Eva Jelínková, Jan Kára, Jan Kratochvíl, Martin Pergel, Ondřej Suchý, and Tomáš Vyskočil. Clustered planarity: Small clusters in cycles and Eulerian graphs. *J. Graph Algorithms Appl.*, 13(3):379–422, 2009.
16. Eugene L. Lawler. Matroid intersection algorithms. *Mathematical Programming*, 9:31–56, 1975.
17. Roy B. Levow. On Tutte's algebraic approach to the theory of crossing numbers. In *Proceedings of the Third Southeastern Conference on Combinatorics, Graph Theory, and Computing (Florida Atlantic Univ., Boca Raton, Fla., 1972)*, pages 315–314, Boca Raton, Fla., 1972. Florida Atlantic Univ.
18. James Oxley. *Matroid Theory*. Oxford University Press, 2011.
19. János Pach and Géza Tóth. Which crossing number is it anyway? *J. Combin. Theory Ser. B*, 80(2):225–246, 2000.
20. Michael J. Pelsmajer, Marcus Schaefer, and Despina Stasi. Strong Hanani–Tutte on the projective plane. *SIAM Journal on Discrete Mathematics*, 23(3):1317–1323, 2009.
21. Michael J. Pelsmajer, Marcus Schaefer, and Daniel Štefankovič. Removing even crossings. *J. Combin. Theory Ser. B*, 97(4):489–500, 2007.
22. Michael J. Pelsmajer, Marcus Schaefer, and Daniel Štefankovič. Removing even crossings on surfaces. In Jarik Nešetřil Alberto Márquez and Oriol Serra, editors, *European Conference on Combinatorics, Graph Theory and Applications (EuroComb '07)*, Electronic Notes in Discrete Mathematics, pages 85–90. Elsevier, August 2007.
23. Marcus Schaefer. Hanani-Tutte and related results. To appear in Bolyai Memorial Volume.
24. Marcus Schaefer. Toward a theory of planarity: Hanani-tutte and planarity variants. In *Graph Drawing*, volume 7704 of *Lecture Notes in Computer Science*, pages 162–173. Springer Berlin Heidelberg, 2013.
25. William T. Tutte. Toward a theory of crossing numbers. *J. Combin. Theory*, 8:45–53, 1970.
26. Wen Jun Wu. On the planar imbedding of linear graphs. I. *J. Systems Sci. Math. Sci.*, 5(4):290–302, 1985.

## Appendix - Omitted proofs

**Lemma 1**. *Let $(G,T)$ denote a two-clustered bipartite graph in which clusters induce independent sets. If $G$ admits an even drawing then $(G,T)$ is c-planar. Moreover, there exists a clustered embedding of $(G,T)$ with the same rotation system as in the given even clustered drawing of $G$.*

*Proof.* Observe that we can assume that $G = (V,E)$ is connected, since we can redraw connected components separately. Let $A$ and $B$ denote the partition of $V$ corresponding to the clusters of $(G,T)$. Thus, both $A$ and $B$ are independent sets.

By the weak Hanani-Tutte theorem $[3,21]$ we can embed $G$ so that the rotation system is the same as in the initial even drawing of $G$. We show that the obtained embedding can be augmented by adding edges, each connecting either two vertices in $A$ or $B$, such that the resulting drawing is still an embedding, and both $A$ and $B$ induce a connected subgraphs in the resulting graph.

Let $E'$ denote a smallest set of edges that can be added to the embedding of $G$, such that the resulting drawing is still an embedding, minimizing the number of connected components of $G'[A]$ and $G'[B]$, where $G' = (V, E \cup E')$. If both $G'[A]$ and $G'[B]$ are connected we are done. Otherwise, there exists a cycle $C$, let's say in $G'[A]$, separating a pair of vertices of $B$. Note that all the edges of $C$ belongs to $E'$. Let $e$ denote an arbitrary edges of $C$. The set $E' - e$ contradicts the choice of $E'$.

Hence, the set $E'$ induces a tree in both $A$ and $B$. By taking a small neighborhood of such a tree in $A$ and $B$, respectively, as a cluster the claim follows.

**Lemma 3**. *An embedded flat clustered graph $(G,T)$, all of whose faces are incident to at most five vertices, can be augmented by adding edges into an embedded flat clustered graph $(G',T')$ such that $(G,T)$ is c-planar if and only if $(G',T')$ is c-planar, and the following holds for $(G',T')$. If $(G',T')$ is c-planar then $(G',T')$ has a saturator $F$ whose edges can be embedded so that at most one edge of $F$ is inside each face of $G'$.*

*Proof.* First we assume that $G$ is vertex 2-connected, in which case we will show that $G = G'$. Take a minimal saturator $F$ of $G$. $F$ can contain only edges between vertices belonging to the same cluster. Suppose by contradiction that $F$ has two edges inside a face. This is only possible if the face is a 5-face, i.e. a pentagon, and the two edges must meet at a vertex. Denote this vertex by $v$ and the two edges by $av$ and $bv$. Clearly, $a, b$ and $v$ belong to the same cluster $C$. But then we can delete $bv$, as $b$ and $v$ are already connected in $G[C]$ by the $vab$ path, a contradiction. Now, we consider the case when $G$ is not 2-connected. By a simple case analysis we show that we can augment $G$ by adding edges thereby obtaining a required graph $G'$.

First, suppose that $G$ contains a face $f$, whose boundary is connected, and whose facial walk contains $C_4$ and one more vertex $v$. Suppose that there exists a vertex $u$ of $C_4$ belonging to the same cluster $C$ as $v$ but in the different component of $G[C]$ as $v$, such that by adding the edge $uv$ to $G$ we split $f$ into a 3-face $uvz$ and a 5-face. Observe that by adding $uv$ to $G$ we did not change the c-planarity of $(G,T)$. Indeed, if $(G-v, T-v)$ is c-planar, we can add the vertex $v$ to a c-planar embedding of $(G-v, T-v)$ in a close vicinity of $u$ and properly draw the edges $vz$ and $vu$ such that the resulting drawing is a c-planar drawing of $(G \cup \{vu\}, T)$. Note that if such a vertex $u$ does not exist, every minimal saturator of $G$ can contain at most one edge belonging to the interior of $f$.

Next, we consider the case when $G$ contains a face $f$, whose boundary is formed by $C_4$, and one additional vertex $v$ not connected with the rest of the graph. If $v$ is in the same cluster as a vertex $u \in C_4$, then $(G - v, T - v)$ is c-planar if and only if $(G, T)$ is c-planar. Thus, we are done in this case. Otherwise, the required property of a minimal saturator holds $f$.

If $G$ has a face $f$, whose boundary is connected, incident to more than three vertices, whose facial walk contains $C_3$, we proceed as follows. If there exists a vertex $v$ incident to all the vertices incident to $f$, we can augment $G$ by adding the edges inside $f$ between consecutive neighbors of $v$ in its rotation whenever they belong to different components of the same cluster. It is easy to see that the required property of a minimal saturator holds for newly created faces in the obtained clustered embedded graph.

Otherwise, it is a very simple, but somewhat tiring, case analysis to see that we can augment $G$ by adding edges inside $f$ such that in the obtained embedded clustered graph no minimal saturator has an edge inside $f$.

If $G$ has a face $f$, whose boundary is formed by $C_3$, and two additional vertices or an edge not connected with the rest of the graph we can again easily augment $G$ by adding edges inside $f$ such that in the obtained embedded clustered graph no minimal saturator has an edge inside $f$.

Similarly we can proceed if $G$ has a face whose facial walk does not contain a cycle, in which case the graph $G$ can have at most 5 vertices.