Left Compressed Shadows

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In this article we present a new proof for the forty years old Kruskal-Katona theorem. This was first proved by J. B. Kruskal in 1963 [6], and later, independently by G. Katona in 1966 [5]. Since then several new proofs were made, the last by R. Ahlswede et. al in 2003 [1]. The shifting technique that we use was first applied in an article of P. Erdős, C. Ko and R. Rado in 1961 [2]. The connections between shifts and shadows was discovered by G. Katona in 1964 [4], this is the method we improved in this article.

I would like to thank G. Katona his advices.

The problem that we are trying to solve is the following: On a finite set S we have a k-uniform family of sets $\mathcal{F}$. (So each element of $\mathcal{F}$ has cardinality k.) Let the *shadow* of a set F or a family of sets $\mathcal{F}$ be the following family of sets:

$$\sigma(F) := \{G : G \subset F, |G| = k - 1\}$$
$$\sigma(\mathcal{F}) := \{\sigma(F) : F \in \mathcal{F}\}$$

The question is at least how big the shadow ($|\sigma(\mathcal{F})|$) has to be if the size of the family ($|\mathcal{F}|$) is fixed. Before we state the main theorem, we introduce some notations to enhance our vocabulary.

Let us fix an arbitrary ordering $<$ of the elements of the set S. Let us denote the smallest element of the set H by $min(H)$ and the largest by $max(H)$. The ordering $<$ can be naturally extended to the subsets of S that have the same cardinality: $F < G \Leftrightarrow max(F \setminus G) < max(G \setminus F)$. We denote the smallest element of the family $\mathcal{H}$ by $Min(\mathcal{H})$ and the largest by $Max(\mathcal{H})$. Now we can extend again this ordering to the subsets of $\mathcal{P}(S)$ that have the

same cardinality and are k-uniform: $\mathcal{F} < \mathcal{G} \Leftrightarrow Max(\mathcal{F} \setminus \mathcal{G}) < Max(\mathcal{G} \setminus \mathcal{F})$. we are going to denote the k element subsets of a set H by $\binom{H}{k}$. Now we can ask our original question using this language:
At least how big is the shadow of an element of $\binom{\binom{S}{k}}{|\mathcal{F}|}$?

Theorem. *(Kruskal-Katona)* $|\sigma(\mathcal{F})| \geq |\sigma(Min\binom{\binom{S}{k}}{|\mathcal{F}|})|$.

In other words, the theorem claims that among the k-uniform, fixed cardinality families the smallest one (considering the above defined ordering) has a shadow thats size is minimal. Computing the size of this shadow we get the following lower bound:

Corollary. *(Kruskal-Katona) Let* $|\mathcal{F}| = \binom{a_k}{k} + \ldots + \binom{a_t}{t}$ *where* $a_k > \ldots > a_t \geq t \geq 1$. *(It is well-known that every positive integer can be written in such form.) In this case* $|\sigma(\mathcal{F})| \geq \binom{a_k}{k-1} + \ldots + \binom{a_t}{t-1}$.

Proof. This follows from the previous theorem and from the easy-to-see $|\sigma(Min\binom{\binom{S}{k}}{|\mathcal{F}|})| = \binom{a_k}{k-1} + \ldots + \binom{a_t}{t-1}$ equality. □

The key of the proof of the theorem is an operation called *left-shift.* (Earlier the operation that we are going to call 1-*left-shift* was called left-shift.)

Definition. *Let* $\mathcal{F} \subseteq \binom{S}{k}$, $X, Y \subseteq S$, $|X| = |Y| \geq 0$, $X \cap Y = \emptyset$, $X > Y$. *The* (X,Y)*-left-shift of* $\mathcal{F}$ *is*

$$\tau_{X,Y}(F) := \begin{cases} F \setminus X \cup Y & \textit{if } X \subseteq F \textit{ and } Y \cap F = \emptyset \\ F & \textit{otherwise} \end{cases},$$

$$\tau_{X,Y}(\mathcal{F}) := \{\tau_{X,Y}(F) : F \in \mathcal{F}, \tau_{X,Y}(F) \notin \mathcal{F}\} \cup \{F : F \in \mathcal{F}, \tau_{X,Y}(F) \in \mathcal{F}\}.$$

A left-shift is an l-left-shift *if* $|X| = l$.
A family $\mathcal{F}$ *is* l-left-compressed *if for any* τ l*-left-shift* $\tau(\mathcal{F}) = \mathcal{F}$.

(If $l = 0$, nothing happens, this was only introduced to simplify notations. Note that every family is 0-left-compressed.)

So the left-shift of a family is another family of the same cardinality with elements of the same size (hence $\tau_{X,Y} : \binom{\binom{S}{k}}{f} \to \binom{\binom{S}{k}}{f}$ function where k and f are arbitrary). Further on because of $X > Y$ we have $\tau_{X,Y}(\mathcal{F}) \leq \mathcal{F}$. We can also observe that if $\mathcal{F}$ is not minimal in $\binom{\binom{S}{k}}{|\mathcal{F}|}$ considering our ordering, then there exists a suitable $\tau_{X,Y}$ left-shift that does not leave $\mathcal{F}$ fix; to show such a left-shift, we only have to guarantee that an element of $\mathcal{F}$ will perish during the left-shift: Let $\mathcal{G} \in \binom{\binom{S}{k}}{|\mathcal{F}|}$, $\mathcal{G} < \mathcal{F}$. Because of the definition of the ordering there exists $G \in \mathcal{G} \setminus \mathcal{F}$, $F \in \mathcal{F} \setminus \mathcal{G}$ for which $G < F$. Now $F \notin \tau_{F\setminus G, G\setminus F}(\mathcal{F})$, therefore $\mathcal{F}$ does not stay fix during this left-shift. Because of our first remark it can only decrease (considering our ordering). Therefore we have proved that there is only one family (thats cardinality is f and that is k-uniform) that is l-left-compressed for every $1 \leq l \leq k$ and it is the one that is the smallest considering our ordering.

Now we are going to show that during an l-left-shift the shadow can never increase if our family was $(l-1)$-left-compressed before the left-shift.

Lemma. *If $\mathcal{F}$ is $(l-1)$-left-compressed $(1 \leq l \leq k)$ and $X, Y \in \binom{S}{l}$, $X \cap Y = \emptyset$, $max(X) > max(Y)$, then $|\sigma(\tau_{X,Y}(\mathcal{F}))| \leq |\sigma(\mathcal{F})|$.*

Proof. $\mathcal{B} := \sigma(\tau_{X,Y}(\mathcal{F})) \setminus \sigma(\mathcal{F})$, so it denotes the shades that were created during the left-shift. $\mathcal{A} := \sigma(\mathcal{F}) \setminus \sigma(\tau_{X,Y}(\mathcal{F}))$, the shades that were eliminated during the left-shift. We need to show a $\mathcal{B} \to \mathcal{A}$ injective function φ, this would imply that at most as many shades were created as many were eliminated, hence the shadow of the family did not increase. To every shade $B \in \mathcal{B}$ we will associate a $\varphi(B) \in \mathcal{A}$.

We know that for every $T \in \tau_{X,Y}(\mathcal{F}) \setminus \mathcal{F}$ we have $X \cap T = \emptyset$ and $Y \subseteq T$ because during the shift from the ancestor of T we left X and added Y to it. For every $B \in \mathcal{B}$ there is a $T \in \tau_{X,Y}(\mathcal{F}) \setminus \mathcal{F}$ such that $B = \sigma(T)$, so B is the shade of T. This implies for every $B \in \mathcal{B}$ that $B \cap X = \emptyset$, $|B \cap Y| \geq l-1$.

For the sake of simplicity we write $\{z\}$ instead of z in case of sets with only one element. Let $K = B \setminus Y$.

Claim. $|B \cap Y| = l$.

Proof. Let us suppose that it is not true and $Y \setminus B = y$. Now B can only be the shade of $K \cup Y \in \tau_{X,Y}(\mathcal{F}) \setminus \mathcal{F}$. Therefore the ancestor of $K \cup Y$ was in the family before the shift, so $K \cup X \in \mathcal{F}$. But in this case $F := \tau_{X\setminus min(X), Y\setminus y}(K \cup X) \in \mathcal{F}$ because the original family was $(l-1)$-left-compressed. (Note that $\tau_{X\setminus min(X), Y\setminus y}$ is indeed an $(l-1)$-left-shift because $X \setminus min(X) > Y \setminus y$ follows from $X > Y$.) Now $B \in \sigma(F)$ contradicts $B \in \mathcal{B}$, this completes our indirect proof. □

The claim obviously implies $B = K \cup Y$. Now our function φ can be defined, $A := \varphi(B) := K \cup X$. ($K \cap X = \emptyset$, because $B \cap X = \emptyset$.) This is injective because B determines K and thus $K \cup X$ as well. $B \in \mathcal{B}$ implies that the set thats shadow contains B, was created during the shift, and this implies that the shadow of the ancestor of this set contains A, therefore $A \in \sigma(\mathcal{F})$. We need to show that $A \in \mathcal{A}$, so the only possibly problem is if $A \in \sigma(\tau_{X,Y}(\mathcal{F}))$, that means $A = \sigma(U)$ where $U \in \tau_{X,Y}(\mathcal{F})$. Now obviously $U \in \mathcal{F}$, in other words U was not created during the shift because $X \subseteq A \subseteq U$. We distinguish two cases:

1. case: $U = A \cup z$ where $z \notin Y$. This implies $\tau_{X,Y}(U) \in \mathcal{F}$, otherwise U would be eliminated during the left-shift. But in this case $B \in \sigma(\tau_{X,Y}(U))$ contradicts $B \in \mathcal{B}$.

2. case: $U = A \cup y$ where $y \in Y$. This implies $F := \tau_{X \setminus min(X), Y \setminus y}(U) \in \mathcal{F}$ because of the $(l-1)$-left-compressedness of the original family. Here $F = K \cup Y \cup min(X)$, thus $B \in \sigma(F)$ but this contradicts $B \in \mathcal{B}$.

This completes the proof of the lemma. □

Now we can easily prove the theorem: Let us choose a family $\mathcal{F}$ thats shadow is minimal and is the smallest (considering our ordering) among these families. Let us suppose indirectly that $\mathcal{F} \neq Min\binom{\binom{S}{k}}{|\mathcal{F}|}$. We know that there exists an $1 \leq l \leq k$ such that $\mathcal{F}$ is not l-left-compressed, so there exist $X, Y \in \binom{S}{l}$, $X \cap Y = \emptyset$, $max(X) > max(Y)$ such that $\tau_{X,Y}(\mathcal{F}) < \mathcal{F}$. Let us apply the lemma for the smallest l with this desired property (the minimality ensures the condition of $(l-1)$-left-compressedness). Applying the shift $\tau_{X,Y}$ the shadow does not increase while the family becomes smaller (considering our ordering), thus it could not have been the smallest before the shift. $\surd$

References

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