

# Convex polygons are cover-decomposable

Dömötör Pálvölgyi\*

Géza Tóth†

## Abstract

We show that for any open convex polygon  $P$  there is a constant  $k(P)$  such that any  $k(P)$ -fold covering of the plane with translates of  $P$  can be decomposed into two coverings.

## 1 Introduction

Let  $\mathcal{P} = \{ P_i \mid i \in I \}$  be a collection of planar sets. It is a  $k$ -fold covering if every point in the plane is contained in at least  $k$  members of  $\mathcal{P}$ . A 1-fold covering is simply called a covering.

**Definition.** A planar set  $P$  is said to be cover-decomposable if the following holds. There exists a constant  $k = k(P)$  such that every  $k$ -fold covering of the plane with translates of  $P$  can be decomposed into two coverings. J. Pach proposed the problem of determining all cover-decomposable sets in 1980. For related problems, conjectures, see [BMP05], Chapter 2.1.

**Conjecture. (J. Pach)** *All planar convex sets are cover-decomposable.*

This conjecture has been verified in three special cases.

**Theorem A.** (i) [P86] *Every centrally symmetric open convex polygon is cover-decomposable.*

(ii) [MP86] *The open unit disc is cover-decomposable.*

(iii) [TT07] *Every open triangle is cover-decomposable.*

In this note we verify the conjecture for open convex polygons.

**Theorem 1.** *Every open convex polygon is cover-decomposable.*

Just like in [P86] and in [TT07], we formulate and solve the problem in its dual form. That is, suppose  $P$  is a polygon of  $n$  vertices and we have a collection  $\mathcal{P} = \{ P_i \mid i \in I \}$  of translates of  $P$ . Let  $O_i$  be the center of gravity of  $P_i$ . The collection  $\mathcal{P}$  is a  $k$ -fold covering of the plane if and only if every translate of  $\bar{P}$ , the reflection of  $P$  through the origin, contains at least  $k$  points of the collection  $\mathcal{O} = \{ O_i \mid i \in I \}$ .

---

\*ELTE, Budapest and EPFL, Lausanne. Supported by OTKA NK 67867

†Rényi Institute, Hungarian Academy of Sciences, Budapest. Supported by OTKA T 038397 and T 046246

The collection  $\mathcal{P} = \{ P_i \mid i \in I \}$  can be decomposed into two coverings if and only if the set  $\mathcal{O} = \{ O_i \mid i \in I \}$  can be colored with two colors, such that every translate of  $\bar{P}$  contains a point of both colors.

Divide the plane into small regions, say, squares, such that each square contains at most one vertex of any translate of  $\bar{P}$ . If a translate of  $\bar{P}$  contains sufficiently many points of  $\mathcal{O}$ , then it contains many points of  $\mathcal{O}$  in one of the little squares. We color the points of  $\mathcal{O}$  separately in each of the squares. If we concentrate on points in just one of the little squares, then instead of translates of  $\bar{P}$  we can consider translates of  $n$  different wedges, corresponding to the  $n$  vertices of  $P$ .

In Section 2 we prove some results about coloring point sets with respect to translates of wedges. In Section 3 we formulate the problem precisely in the dual version, and apply the results of Section 2 to prove Theorem 1.

## 2 Preparation

Suppose we have two halflines,  $e$  and  $f$ , both of endpoint  $O$ . Then they divide the plane into two parts,  $W_1$   $W_2$ , which we call *wedges*. A *closed wedge* contains its boundary, an *open wedge* does not. Point  $O$ , where the two boundary lines meet, is called the *apex* of the wedges. The *angle* of a wedge is the angle between its two boundary halflines, measured inside the wedge. That is, the sum of the angles of  $W_1$  and  $W_2$  is  $2\pi$ . Now let  $W$  be a wedge, and  $X$  be a point in the plane. A translate of  $W$  such that its apex is at  $X$ , is denoted by  $W(X)$ . More generally, for points  $X_1, X_2, \dots, X_k$ ,  $W(X_1, X_2, \dots, X_k)$  denotes the *minimal* translate of  $W$  (for containment) whose closure contains  $X_1, X_2, \dots, X_k$ . The set of all translates of  $W$  is denoted by  $Tr^W$ . Let  $-W$  denote the reflection of  $W$  about the origin.

**Definition 1.** Suppose that  $\mathcal{W} = \{ W_i \mid i \in I \}$  is a collection of wedges.  $\mathcal{W}$  is said to be *non-conflicting* or simply NC, if there is a constant  $k$  with the following property. Any finite set of points  $S$  can be colored with two colors such that any translate of a wedge  $W \in \mathcal{W}$  that contains at least  $k$  points of  $S$ , contains a point of both colors.

For any fixed  $\mathcal{W}$  and  $S$ , we can and will assume without loss of generality that the points of  $S$  are in general position with respect to  $\mathcal{W}$ , that is, they do not determine a line parallel to the boundary line of any  $W \in \mathcal{W}$ . Indeed, if there are two such points, slightly perturb the points. It is easy to see that every subset of the original point set that can be cut off by a translate of a  $W \in \mathcal{W}$ , can also be cut off the perturbed point set.

First suppose that we have only one open wedge, that is,  $\mathcal{W} = \{W\}$ .

**Lemma 1.** *A single wedge is NC.*

**Proof of Lemma 1.** Let  $S$  be a finite point set and  $W$  a wedge. We prove the statement with  $k = 3$ , that is,  $S$  can be colored with two colors such that any translate of  $W$  that contains at least 3 points of  $S$ , contains a point of both colors. Suppose first that the angle of  $W$  is at least  $\pi$ . Then  $W$  is the union of two halfplanes,  $A$  and  $B$ . Take the translate of  $A$  (resp.  $B$ ) that contains exactly two points of  $S$ , say,  $A_1$  and  $A_2$  (resp.  $B_1$  and  $B_2$ ). There might be coincidences between  $A_1, A_2$  and  $B_1, B_2$ ,

therefore the set  $\{A_1, A_2, B_1, B_2\}$  contains two, three, or four different points. But in any case, we can color  $\{A_1, A_2, B_1, B_2\}$  such that  $A_1$  and  $A_2$  (resp.  $B_1$  and  $B_2$ ) are of different colors. Now, if a translate of  $W$  contains three points, it contains either  $A_1$  and  $A_2$ , or  $B_1$  and  $B_2$ , and we are done.

Suppose now that the angle of  $W$  is less than  $\pi$ . We give two proofs in this case, since we will apply the ideas of both proofs later. In fact, we show that in this case the NC property holds with  $k = 2$ . We can assume that the positive  $x$ -axis is in  $W$ , and that no two points have the same  $y$ -coordinate. Both of these can be achieved by an appropriate rotation. We say that  $X <_y Y$  if the  $y$ -coordinate of  $X$  is smaller than the  $y$ -coordinate of  $Y$ . This ordering is called the  $y$ -ordering. A subset  $I$  of  $S$  is an *interval* of  $S$  if  $\forall X <_y Y <_y Z \in S : X, Z \in I \rightarrow Y \in I$ .

The *boundary* of  $S$  with respect to  $W$ ,  $Bd^W(S) = \{X \in P : W(X) \cap S = \emptyset\}$ . Note that a translate of  $W$  always intersects the boundary in an interval. For each  $X \in Bd^W(S)$  the *shadow* of  $X$  is  $Sh^W(X) = \{Y \in S : W(Y) \cap Bd^W(S) = X\}$ . Observe that  $\forall X, Y \in Bd^W(S) : Sh^W(X) \cap Sh^W(Y) = \emptyset$ .

For the first proof, color the points of the boundary alternating, according to the order  $<_y$ . For every boundary point  $X$ , color every point in the shadow of  $X$  to the other color than  $X$ . Color the rest of the points arbitrarily. Any translate of  $W$  that contains at least two points, contains one or two boundary points. If it contains one boundary point, then the other point is in its shadow, so they have different colors. If it contains two boundary points, then they are consecutive points according to the  $y$ -order, so they have different colors again. This finishes the first proof.

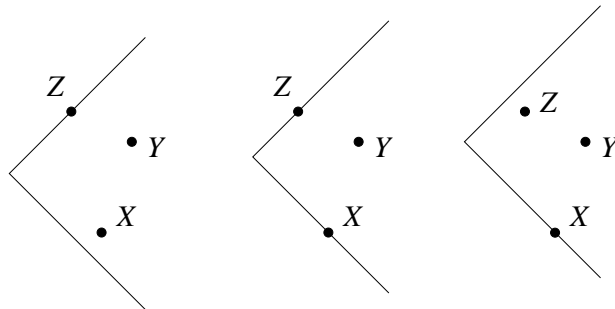


Figure 1:  $Z$  replaces  $X$  in  $W(2; y)$ .

For the second proof, for any fixed  $y$ , let  $W(2; y)$  be the translate of  $W$  which (1) contains at most two points of  $S$ , (2) its apex has  $y$ -coordinate  $y$ , and (3) its apex has minimal  $x$ -coordinate. It is easy to see that for any  $y$ ,  $W(2; y)$  is uniquely defined. Examine, how  $W(2; y)$  changes as  $y$  runs over the real numbers. If  $y$  is very small (smaller than the  $y$ -coordinate of the points of  $S$ ), then  $W(2; y)$  contains two points, say  $X$  and  $Y$ , and one more on its boundary. As we increase  $y$ , the apex of  $W(2; y)$  changes continuously. How can the set  $\{X, Y\}$ , of the two points in  $W(2; y)$  change? For a certain value of  $y$ , one of them, say,  $X$ , moves to the boundary. At this point we have  $Y$  inside, and two points,  $X$ , and  $Z$  on the boundary. If we slightly further increase  $y$ , then  $Z$  replaces  $X$ , that is,  $Y$  and  $Z$  will be in  $W(2; y)$  (see Figure 1). As  $y$  increases to infinity, the set  $\{Z, Y\}$ , could change several times, but the

same way. Define a graph whose vertices are the points of  $S$ , and two vertices,  $U$  and  $V$  are connected iff  $V$  replaced  $U$  during the procedure. We get two paths,  $P_1$  and  $P_2$ . The pair  $(P_1, P_2)$  is called the *path decomposition of  $S$  with respect to  $W$ , of order two*.

On Figure 2, two examples are given. The vertices of  $P_1$  are  $X_1, X_2, \dots$ , and the vertices of  $P_2$  are  $Y_1, Y_2, \dots$ . On both examples, the translates of  $W$  are given in the positions when some point is replacing another.

Color the vertices of  $P_1$  red, the vertices of  $P_2$  blue. Observe that each translate of  $W$  that contains at least two points, contains at least one vertex of both  $P_1$  and  $P_2$ . This completes the second proof.  $\square$

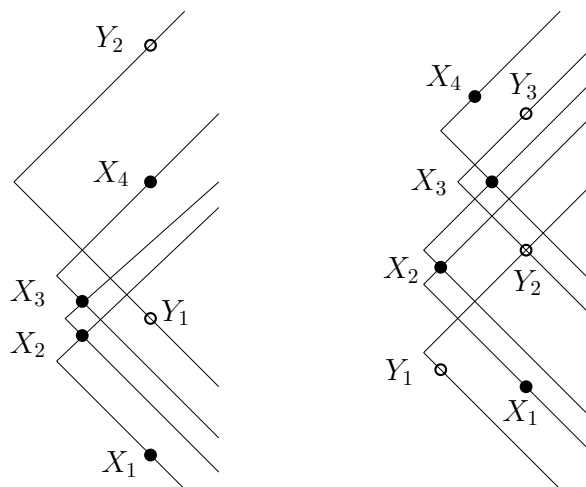


Figure 2: Path decompositions of order two.  $P_1 = X_1X_2\dots$ ,  $P_2 = Y_1Y_2\dots$

Now we can turn to the case when we have translates of *two* wedges at the same time. We distinguish several cases according to the relative position of the two wedges,  $V$  and  $W$ .

**Type 1 (Big):** One of the wedges has angle at least  $\pi$ .

For the other cases, we can assume without loss of generality that  $W$  contains the positive  $x$ -axis. Extend the boundary halflines of  $W$  to lines, they divide the plane into four parts, Upper, Lower, Left, and Right, which is  $W$  itself. See Figure 3.

**Type 2 (Halfplane):** One side of  $V$  is in Right and the other one is in Left. That is, the union of the wedges cover a halfplane. See Figure 4.

**Type 3 (Contain):** Either (i) one side of  $V$  is in Upper, the other one is in Lower, or (ii) both sides are in Right or (iii) both sides are in Left. See Figure 5.

**Type 4. (Hard):** One side of  $V$  is in Left and the other one is in Upper or Lower. This will be the hardest case. See Figure 6.

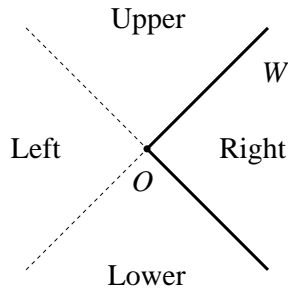


Figure 3: Wedge  $W$

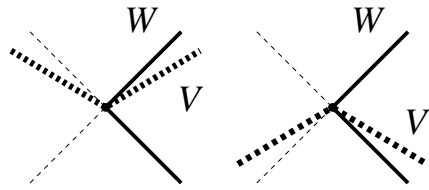


Figure 4: Type 2 (Halfplane)

**Type 5. (Special):** Either (i) one side of  $V$  is in Right and the other one is in Upper or Lower, or (ii) both sides are in Upper, or (iii) both sides are in Lower. That is, the union of the wedges is in an open halfplane whose boundary contains the origin, but none of them contain the other. See Figure 7.

It is not hard to see that there are no other possibilities. Observe that two wedges corresponding to the vertices of a convex polygon cannot be of type 1 (Big) or of type 5 (Special). This will be used in the proof of Theorem 1.

It is shown in [P08] that if  $\mathcal{W} = \{V, W\}$  is a set of two wedges of type 5 (Special), then  $\mathcal{W}$  is *not* NC. In a series of Lemmas we show that all other pairs are NC.

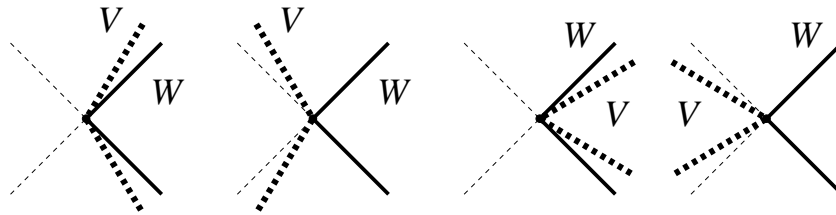


Figure 5: Type 3 (Contain)

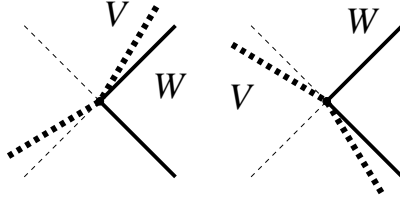


Figure 6: Type 4 (Hard)

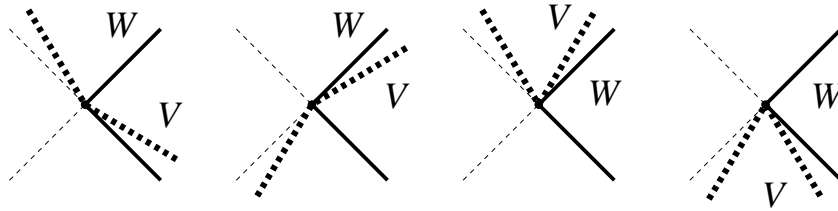


Figure 7: Type 5 (Special)

**Lemma 2.** *Let  $\mathcal{W} = \{V, W\}$  be a set of two wedges, of type 3 (Contain). Then  $\mathcal{W}$  is NC.*

Suppose that  $U$  is a wedge that contains the positive  $x$ -axis and has angle less than  $\pi$ . For any  $k > 0$ , and point set  $S$ , let  $Tr_k^U(S)$  be the set of translates of  $U$  that contain exactly  $k$  points of  $S$ .

Just like in the proof of Lemma 1, for any fixed  $y$ , and  $k > 0$ , let  $U(k; y)$  be the translate of  $U$  which (1) contains at most  $k$  points of  $S$ , (2) its apex has  $y$ -coordinate  $y$ , and (3) its apex has minimal  $x$ -coordinate. If  $y$  is very small, then  $U(k; y)$  contains  $k$  points, say  $X_1, X_2, \dots, X_k$ , and one more on its boundary. As we increase  $y$ , the apex of  $U(k; y)$  changes continuously. For a certain value of  $y$ , one of  $X_1, X_2, \dots, X_k$ , say,  $X_1$ , moves to the boundary. At this point we have  $X_2, \dots, X_k$  inside, and two points,  $X_1$ , and  $X'_1$  on the boundary. If we slightly further increase  $y$ , then  $X'_1$  replaces  $X_1$ , that is,  $X'_1$  and  $X_2, \dots, X_k$  will be in  $U(k; y)$ . As  $y$  increases to infinity, some other points could be replaced similarly. Define a graph whose vertices are the points of  $S$ , and two vertices,  $X$  and  $Y$  are connected iff  $X$  replaced  $Y$  during the procedure. We get  $k$  paths,  $P_1^U, P_2^U, \dots, P_k^U$ . Each translate of  $U$  that contains at least  $k$  points of  $S$ , contains at least one vertex of each of  $P_1^U, P_2^U, \dots, P_k^U$ . The  $k$ -tuple  $(P_1^U, P_2^U, \dots, P_k^U)$  is called the *path decomposition of  $S$  with respect to  $W$ , of order  $k$* .

**Proof of Lemma 2.** We can assume that  $W \supset V$  or  $W \supset -V$  and  $W$  contains the positive  $x$ -axis. Let  $(P_1^W, P_2^W, \dots, P_k^W)$  be the path decomposition of  $S$  with respect to  $W$ , of order  $k$ .

Observe that any translate of  $V$  intersects any  $P_i^W$  in an interval of it. Indeed, if  $X_1 <_y X_2 <_y X_3 \in P_i^W$ , then  $X_2 \in W(X_1, X_3) \cap -W(X_1, X_3)$ , which is a subset of  $V(X_1, X_3) \cap -V(X_1, X_3)$ . See Figure 8.

We show that we can color the points of  $S$  with red and blue such that any translate of  $W$  which

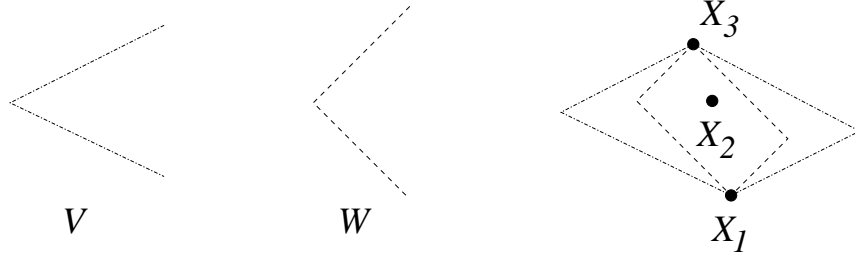


Figure 8:  $W(X_1, X_3) \cap -W(X_1, X_3) \subset V(X_1, X_3) \cap -V(X_1, X_3)$ .

contains at least 4 points, and any translate of  $V$  which contains at least 14 points, contains points of both colors. Consider  $(P_1^W, P_2^W, P_3^W, P_4^W)$ , the path decomposition of  $S$  with respect to  $W$ , of order 4. We color  $P_1^W$  and  $P_2^W$  such that every  $W' \in Tr_4^W(S)$  contains a blue point of them, and every  $V' \in Tr_7^V(P_1^W \cup P_2^W)$  contains points of both colors. Similarly, we color  $P_3^W$  and  $P_4^W$  such that every  $W' \in Tr_4^W(S)$  contains a red point of them, and every  $V' \in Tr_7^V(P_3^W \cup P_4^W)$  contains points of both colors. Finally, we color the rest of the points  $R = S \setminus (P_1^W \cup P_2^W \cup P_3^W \cup P_4^W)$  such that every  $V' \in Tr_2^V(R)$  contains points of both colors.

Recall that for any  $W' \in Tr_4^W(S)$ ,  $|W' \cap P_1^W| = |W' \cap P_2^W| = |W' \cap P_3^W| = |W' \cap P_4^W| = 1$ . For any  $X \in P_1^W, Y \in P_2^W$ , if there is a  $W' \in Tr_4^W(S)$  with  $W' \cap P_1^W = \{X\}$  and  $W' \cap P_2^W = \{Y\}$ , then we say that  $X$  and  $Y$  are *friends*. If  $X$  (resp.  $Y$ ) has only one friend  $Y$  (resp.  $X$ ), then we call it a *fan* (of  $Y$ , resp. of  $X$ ). If  $X$  or  $Y$  has at least one fan, then we say that it is a *star*. Those points that are neither fans, nor stars are called *regular*.

For an example, see Figure 2. On the left figure,  $Y_1$  is a star, its fans are  $X_2$  and  $X_3$ , the other points are regular. On the right,  $Y_2$  is a star, its fan is  $X_2$ , the other points are regular.

Suppose first that all points of  $P_1^W$  and  $P_2^W$  are regular. Color every third point of  $P_1^W$ , red and the others blue. In  $P_2^W$ , color the friends of the red points blue, and color the rest of the points of  $P_2^W$  (every third) red. For any  $W' \in Tr_4^W$ ,  $W' \cap P_1^W$  and  $W' \cap P_2^W$  are friends, therefore, at least one of them is blue. On the other hand, any  $V' \in Tr_7^V(P_1^W \cup P_2^W)$  contains three consecutive points of  $P_1^W$  or  $P_2^W$ , and they have both colors.

Suppose now that not all the points of  $P_1^W$  and  $P_2^W$  are regular. Color all stars blue. The first and last friend of a star, in the  $y$ -ordering, is either a star or a regular vertex, the others are fans. Color the friends of each star alternatingly, according to the  $y$ -ordering, starting with blue, except the last two friends; color the last one blue, the previous one red. The so far uncolored regular points of  $P_1^W$  and  $P_2^W$  form pairs of intervals. We color each such pair of interval the same way as we did in the all-regular case, coloring the first point of each pair of intervals red. See Figure 9.

Clearly, if  $W' \in Tr_4^W$  then it contains at least one blue point of  $P_1^W \cup P_2^W$ . If  $V' \in Tr_7^V(P_1^W \cup P_2^W)$ , then it contains four consecutive points of  $P_1^W$  or  $P_2^W$ , say,  $X_1, X_2, X_3, X_4$ , in  $P_1^W$ . If  $X <_y Y <_y Z \in P_1^W \cap V'$  and  $Y$  is a star, then  $V'$  must contain all fans of  $Y$  as well. Indeed, the fans of  $Y$  are

in  $W(X, Z) \setminus (W(X) \cup W(Z))$ , and by our earlier observations, this is in  $V(X, Z) \subset V'$ . So, if either  $X_2$  or  $X_3$  is a star, then  $V'$  contains a red point, since every star has a red fan. Since the star itself is blue, we are done in this case. If  $X_1, X_2, X_3, X_4$  contains three consecutive regular vertices then we are done again, by the coloring rule for the regular intervals. So we are left with the case when  $X_1$  and  $X_4$  are stars,  $X_2$  and  $X_3$  are regular. But in this case  $V'$  also contains the common friend  $Y$  of  $X_2$  and  $X_3$  in  $P_2^W$ , which is also a regular vertex. By the coloring rule for the regular intervals, one of  $Y, X_2$  and  $X_3$  is red, the other two are blue, so we are done.

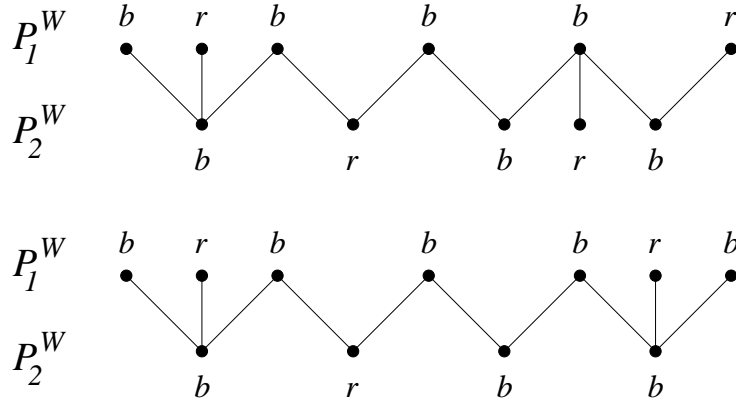


Figure 9: Two examples of coloring of  $P_1^W \cup P_2^W$ . Friends are connected by edges.

For  $P_3^W \cup P_4^W$  we use the same coloring rule as for  $P_1^W \cup P_2^W$  but we switch the roles of the colors. So any  $W' \in Tr_4^W$  contains at least one red point of  $P_3^W \cup P_4^W$  and any  $V' \in Tr_7^V(P_3^W \cup P_4^W)$  contains both colors.

Finally, we have to color the rest of the points  $R = S \setminus (P_1^W \cup P_2^W \cup P_3^W \cup P_4^W)$  such that every  $V' \in Tr_2^V(R)$  contains points of both colors. This can be achieved by the first proof of Lemma 1.

Now any  $W' \in Tr_4^W$  contains at least one blue and at least one red point. If  $V' \in Tr_{14}^V$ , then either it contains at least two points of  $R = P \setminus (P_1^W \cup P_2^W \cup P_3^W \cup P_4^W)$ , or at least seven points of  $P_1^W \cup P_2^W$ , or at least seven points of  $P_3^W \cup P_4^W$ , and in all cases it contains points of both colors. This completes the proof of Lemma 2.  $\square$

**Definition 2.** Suppose that  $\mathcal{W} = \{V, W\}$  is a pair of wedges.  $\mathcal{W}$  is said to be *asymmetric non-conflicting* or simply *ANC*, if there is a constant  $k$  with the following property. Any finite set of points  $S$  can be colored with red and blue such that any translate of  $V$  that contains at least  $k$  points of  $S$ , contains a red point, and any translate of  $W$  that contains at least  $k$  points of  $S$ , contains a blue point.

The next technical result allows us to simplify all following proofs.

**Lemma 3.** *If a pair of wedges is not of type Special, and ANC, then it is also NC.*



**Proof of Lemma 3.** We can assume without loss of generality that  $V$  contains the positive  $x$ -axis, and  $W$  contains either the positive or the negative  $x$ -axis. Suppose that  $\{V, W\}$  is ANC, let  $k > 0$  arbitrary, and let  $S$  be a set of points. First we color  $Bd^V(S)$ . Let  $U$  be a wedge that also contains the positive  $x$ -axis, but has a very small angle. Then translates of  $V$  and translates of  $U$  both intersect  $Bd^V(S)$  in its intervals. Clearly, the pair  $\{U, W\}$  is of type Contain, therefore, by Lemma 2, we can color  $Bd^V(S)$  such that any translate of  $W$ ,  $W' \in Tr_4^W(Bd^V(S))$  and any translate of  $U$ ,  $U' \in Tr_{14}^U(Bd^V(S))$  contains both colors. But then any translate of  $V$ ,  $V' \in Tr_{14}^V(Bd^V(S))$  contains both colors as well.

Now we have to color  $S \setminus Bd^V(S)$ . We divide it into three parts as follows.

$$S_b = \{X \in S \setminus Bd^V(S) \mid \forall Y \in V(X) \cap Bd^V(S), Y \text{ is blue}\},$$

$$S_r = \{X \in S \setminus Bd^V(S) \mid \forall Y \in V(X) \cap Bd^V(S), Y \text{ is red}\},$$

$$S_0 = S \setminus (Bd^V(S) \cup S_b \cup S_r).$$

Any translate  $V' \in Tr^V$  that intersects  $S_b$  in at least one point, must contain at least one blue point, from  $Bd^V(S)$ , so we only have to make sure that it contains a red point too. Similarly, any  $V' \in Tr^V$  that intersects  $S_r$  in at least one point, must contain a red point, and any  $V' \in Tr^V$  that intersects  $S_0$  must contain points of both colors.

Thus, we can simply color  $S_0$  such that any  $W' \in Tr_2^W(S_0)$  contains both colors, which can be done by Lemma 1.

With  $S_b$ , and with  $S_r$ , respectively, we proceed exactly the same way as we did with  $S$  itself, but now we change the roles of  $V$  and  $W$ . We get the (still uncolored) subsets  $S_{b,b}$ ,  $S_{b,r}$ ,  $S_{b,0}$ ,  $S_{r,b}$ ,  $S_{r,r}$ ,  $S_{r,0}$  with the following properties.

- Any translate  $V' \in Tr^V$  or  $W' \in Tr^W$ , that intersects  $S_{b,b}$  (resp.  $S_{r,r}$ ) in at least one point, must contain at least one blue (resp. red) point.
- Any translate  $V' \in Tr^V$  that intersects  $S_{b,r}$  (resp.  $S_{r,b}$ ) contains a blue (resp. red) point, and any translate  $W' \in Tr^W$  that intersects  $S_{b,r}$  (resp.  $S_{r,b}$ ) contains a red (resp. blue) point.
- Any translate  $V' \in Tr^V$  that intersects  $S_{b,0}$  (resp.  $S_{r,0}$ ) contains a blue (resp. red) point, and any translate  $W' \in Tr^W$  that intersects  $S_{b,0}$  (resp.  $S_{r,0}$ ) contains points of both colors.

Color all points of  $S_{b,b}$  and  $S_{b,0}$  red, color all points of  $S_{r,r}$  and  $S_{r,0}$  blue. Finally, color  $S_{b,r}$  using the ANC property of the pair  $(V, W)$ , and similarly, color  $S_{r,b}$  also using the ANC property, but the roles of red and blue switched. Now it is easy to check that in this coloring any translate of  $V$  or  $W$  that contains sufficiently many points of  $S$ , contains a point of both colors.  $\square$

**Remark.** In [P08] it has been proved that if  $\{V, W\}$  is a Special pair, then  $\{V, W\}$  is *not* ANC. So, the following statement holds as well.

**Lemma 3'.** *If a pair of wedges is ANC, then it is also NC.*

**Lemma 4.** *Let  $\mathcal{W} = \{V, W\}$  be a set of two wedges, of type 1 (Big). Then  $\mathcal{W}$  is NC.*

**Proof of Lemma 4.** By Lemma 3, it is enough to show that  $\{V, W\}$  is ANC. Let  $W$  be the wedge whose angle is at least  $\pi$ . Then  $W$  is the union of two halfplanes, say,  $H_1$  and  $H_2$ . Translate both halfplanes such that they contain exactly one point of  $S$ , denote them by  $X_1$  and  $X_2$ , respectively. Note that  $X_1$  may coincide with  $X_2$ . Color  $X_1$  and  $X_2$  red, and all the other points blue. Then any translate of  $W$  that contains at least one point, contains a red point, and any translate of  $V$  that contains at least three points, contains a blue point.  $\square$

**Lemma 5.** *Let  $\mathcal{W} = \{V, W\}$  be a set of two wedges, of type 2 (Halfplane). Then  $\mathcal{W}$  is NC.*

**Proof of Lemma 5.** Again, it is enough to show that they are ANC. Since  $\{V, W\}$  is of type 2 (Halfplane),  $Bd^V(S)$  and  $Bd^W(S)$  have at most one point in common. If  $Bd^V(S)$  and  $Bd^W(S)$  are disjoint, then color  $Bd^V(S)$  blue,  $Bd^W(S)$  red, and the other points arbitrarily. Then any nonempty translate of  $V$  (resp.  $W$ ) contains a blue (resp. red) point.

Otherwise, let  $X$  be their common point. Let  $P = Bd^V(S) \cup Bd^W(S) \setminus X$ , and consider its  $V$ -boundary,  $Bd^V(P)$ , and  $W$ -boundary,  $Bd^W(P)$ . Clearly, each point in  $P = Bd^V(S) \setminus X$  belongs to  $Bd^V(P)$ , and each point in  $P = Bd^W(S) \setminus X$  belongs to  $Bd^W(P)$ .

If  $Bd^V(P)$  and  $Bd^W(P)$  are disjoint, then color  $Bd^V(S)$  blue,  $Bd^W(P)$  and the other points red. Then any nonempty translate of  $V$  contains a blue point. Suppose that we have a translate of  $W$  with two points, both blue. Then it should contain  $X$ , and a point of  $Bd^V(P)$ . But this contradicts our assumption that  $Bd^V(P)$  and  $Bd^W(P)$  are disjoint. So, any translate of  $W$  which contains at least two points of  $S$ , contains a red point.

If  $Bd^V(P)$  and  $Bd^W(P)$  are not disjoint, then they have one point in common, let  $Y$  be their common point. If  $Y$  belongs to  $Bd^W(P)$ , then color  $Bd^V(S)$  blue,  $Bd^W(P)$  and the other points red. Then, by the same argument as before, any nonempty translate of  $V$  contains a blue point, and any translate of  $W$  which contains at least two points of  $S$ , contains a red point. Finally, if  $Y$  belongs to  $Bd^V(P)$ , then we proceed analogously, but the roles of  $V$  and  $W$ , and the colors, are switched.  $\square$

**Lemma 6.** *Let  $\mathcal{W} = \{V, W\}$  be a set of two wedges, of type 4 (Hard). Then  $\mathcal{W}$  is NC.*

**Proof of Lemma 6.** As usual, we only prove that  $\{V, W\}$  is ANC. Assume that  $W$  contains the positive  $x$ -axis. Just like in the definition of type 4 (Hard), extend the boundary halfines of  $W$  to lines, they divide the plane into four parts, Upper, Lower, Left, and Right, which is  $W$  itself. We can assume without loss of generality that  $V$  contains the negative  $x$ -axis, one side of  $V$  is in Upper, and one side is in Left.

Observe that if a translate of  $V$  and a translate of  $W$  intersect each other, then one of them contains the other one's apex.

**Claim 1.** For any point set  $P$  and  $X \in P$ , either  $Bd^V(P \setminus X) \setminus Bd^V(P) = \emptyset$  or  $Bd^W(P \setminus X) \setminus Bd^W(P) = \emptyset$ .

**Proof of Claim 1.** Suppose on the contrary that  $Y \in Bd^V(P \setminus X) \setminus Bd^V(P)$  and  $Z \in Bd^W(P \setminus X) \setminus Bd^W(P)$ . Then  $X \in V(Y)$  and  $X \in W(Z)$ , so  $V(Y)$  and  $W(Z)$  intersect each other, therefore,

one of them contains the other one's apex, say,  $Z \in V(Y)$ . But this is a contradiction, since  $Y$  is a boundary point of  $P \setminus X$ .

Return to the proof of Lemma 6. Color  $Bd^V(S) \setminus Bd^W(S)$  red, and  $Bd^W(S) \setminus Bd^V(S)$  blue, the interior points arbitrarily. Now consider the points of  $Bd^V(S) \cap Bd^W(S)$ . For any  $X \in Bd^V(S) \cap Bd^W(S)$ , if  $Bd^V(S \setminus X) \setminus Bd^V(S) \neq \emptyset$ , then color it red, if  $Bd^W(S \setminus X) \setminus Bd^W(S) \neq \emptyset$ , then color it blue. For each of the remaining points  $Y$  we have  $Bd^V(S \setminus Y) \setminus Bd^V(S) = Bd^W(S \setminus Y) \setminus Bd^W(S) = \emptyset$ . Color each of these points such that they have the *opposite* color than the the previous point of  $Bd^V(S) \cap Bd^W(S)$ , in the  $y$ -ordering.

To prove that this coloring is good, let  $V' \in Tr_2^V$ ,  $V' \cap S = \{X, Y\}$ . If it intersects  $Bd^V(S) \setminus Bd^W(S)$ , we are done. So assume that  $V' \cap Bd^V(S) \subset Bd^V(S) \cap Bd^W(S)$ . Let  $X \in V' \cap Bd^V(S)$ . If  $X$  is red, then by the coloring rule,  $Bd^V(S \setminus X) \setminus Bd^V(S) = \emptyset$ . But then  $Y$  is also a  $V$ -boundary point, so we have  $Y \in Bd^V(S) \cap Bd^W(S)$ . Again we can assume that  $Y$  is red, so  $Bd^V(S \setminus Y) \setminus Bd^V(S) = \emptyset$ . Suppose that  $X <_y Y$ . Since  $V' \cap S = \{X, Y\}$ ,  $X$  and  $Y$  are consecutive points of  $Bd^V(S) \cap Bd^W(S)$ . Now it is not hard to see that  $Bd^W(S \setminus Y) \setminus Bd^W(S) = \emptyset$ . Therefore, by the coloring rule,  $X$  and  $Y$  have different colors. For translates of  $W$  the argument is analogous, with the colors switched.  $\square$

Now we turn to the case when we have more than two wedges.

**Lemma 7.** *For any  $s, t > 0$  integers, there is a number  $f(s, t)$  with the following property.*

*Let  $\mathcal{W} = \{W_1, W_2, \dots, W_t\}$  be a set of  $t$  wedges, such that any pair  $\{W_i, W_j\}$  is NC, and let  $S$  be a set of points. Then  $S$  can be decomposed into  $t$  parts,  $S_1, S_2, \dots, S_t$ , such that for  $i = 1, 2, \dots, t$ , for any translate  $W'_i$  of  $W_i$ , if  $|W'_i \cap S| \geq f(s, t)$  then  $|W'_i \cap S_i| \geq s$ .*

**Proof of Lemma 7.** The existence of  $f(1, 2)$  is equivalent to the property that the corresponding two wedges are ANC. Now we show that  $f(s, 2)$  exists for every  $s$ . Let  $V$  and  $W$  be two wedges that form a NC pair. Let  $P_1^V, P_2^V, \dots, P_{s^2 f(1, 2)}^V$  be the path decomposition of  $S$  of order  $s^2 f(1, 2)$ , with respect to  $V$ . For  $i = 1, 2, \dots, s$ , let

$$H_i = \cup_{j=(i-1)sf(1,2)+1}^{isf(1,2)} P_j^V.$$

For each  $H_i$ , take the  $W$ -path decomposition,  $P_1^W(H_i), \dots, P_{sf(1,2)}^W(H_i)$ , and for  $j = 1, 2, \dots, s$ , let

$$H_i^j = \cup_{k=(j-1)f(1,2)+1}^{jf(1,2)} P_k^W(H_i).$$

For every  $i, j = 1, 2, \dots, s$ , color  $H_i^j$ , such that any translate of  $V$  (resp.  $W$ ) that intersects it in at least  $f(1, 2)$  points, contains at least one red (resp. blue) point of it. This is possible, since the pair  $\{V, W\}$  is ANC.

Consider a translate  $V'$  of  $V$  that contains at least  $s^2 f(1, 2)$  points of  $S$ . For every  $i$ ,  $V'$  intersects  $H_i$  in  $sf(1, 2)$  points, so there is a  $j$  such that it intersects  $H_i^j$  in at least  $f(1, 2)$  points. Therefore,  $V'$  contains at least one red point of  $H_i^j$ , so at least  $s$  red points of  $S$ .

Consider now a translate  $W'$  of  $W$  that contains at least  $s^2 f(1, 2)$  points of  $S$ . There is an  $i$  such that  $W'$  intersects  $H_i$  in at least  $sf(1, 2)$  points. Therefore, it intersects each of  $P_1^W(H_i), \dots, P_{sf(1,2)}^W(H_i)$ ,

in at least one point, so for  $j = 1, 2, \dots, s$ ,  $W'$  intersects  $H_i^j$  in at least  $f(1, 2)$  points. Consequently, it contains at least one blue point of each  $H_i^j$ , so at least  $s$  blue points of  $S$ .

Now let  $s, t > 2$  fixed and suppose that  $f(s', t - 1)$  exists for every  $s'$ . Let  $\{W_1, W_2, \dots, W_t\}$  be our set of wedges, such that any pair of them is NC. Let  $s' = f(s, 2)$ . Partition our point set  $S$  into  $S'_1, S'_2, \dots, S'_{t-1}$  such that for  $i = 1, 2, \dots, t - 1$ , for any translate  $W'_i$  of  $W_i$ , if  $|W'_i \cap S| \geq f(s', t - 1)$  then  $|W'_i \cap S'_i| \geq s' = f(s, 2)$ . For each  $i = 1, 2, \dots, t - 1$ , partition  $S'_i$  into two parts,  $S''_i$  and  $S^t_i$ , such that for any translate  $W'_i$  of  $W_i$ , if  $|W'_i \cap S'_i| \geq f(s, 2)$  then  $|W'_i \cap S''_i| \geq s$ , and for any translate  $W'_t$  of  $W_t$ , if  $|W'_t \cap S'_i| \geq f(s, 2)$  then  $|W'_t \cap S^t_i| \geq s$ . Finally, for  $i = 1, 2, \dots, t - 1$ , let  $S_i = S''_i$  and let  $S_t = \cup_{j=1}^{t-1} S^t_j$ . For  $i = 1, 2, \dots, t - 1$ , any translate  $W'_i$  of  $W_i$ , if  $|W'_i \cap S| \geq f(s', t - 1)$  then  $|W'_i \cap S'_i| \geq s' = f(s, 2)$ , so  $|W'_i \cap S_i| \geq s$ ,

And for any translate  $W'_t$  of  $W_t$ , if  $|W'_t \cap S| \geq f(s', t - 1)$ , then for some  $i = 1, 2, \dots, t - 1$ ,  $|W'_t \cap S'_i| \geq \frac{f(s', t - 1)}{t - 1} \geq f(s, 2)$ , therefore,  $|W'_t \cap S^t_i| \geq s$ , so  $|W'_t \cap S_t| \geq s$ . This concludes the proof of Lemma 7.  $\square$

**Remark.** The proofs of Lemmas 2, 4, 5, and 6 imply that  $f(1, 2) \leq 8$ . Combining it with the proof of Lemma 7 we get the bound  $f(s, t) \leq (8s)^{2^{t-1}}$ .

**Lemma 8.** *A set of wedges  $\mathcal{W} = \{W_1, W_2, \dots, W_t\}$  is NC if and only if any pair  $\{W_i, W_j\}$  is NC.*

**Proof of Lemma 8.** Clearly, if some pair  $\{W_i, W_j\}$  is *not* NC, then the whole set  $\mathcal{W}$  is not NC either. Suppose that every pair  $\{W_i, W_j\}$  is NC. Decompose  $S$  into  $t$  parts  $S_1, S_2, \dots, S_t$  with the property that for  $i = 1, 2, \dots, t$ , for any translate  $W'_i$  of  $W_i$ , if  $|W'_i \cap S| \geq f(3, t)$  then  $|W'_i \cap S_i| \geq 3$ . Then, by Lemma 1, each  $S_i$  can be colored with red and blue such that if  $|W'_i \cap S_i| \geq 3$  then  $W'_i$  contains points of both colors. So this coloring of  $S$  has the property that for  $i = 1, 2, \dots, t$ , for any translate  $W'_i$  of  $W_i$ , if  $|W'_i \cap S| \geq f(3, t)$  then it contains points of both colors.  $\square$

### 3 Proof of Theorem 1.

Suppose that  $P$  is an open convex polygon of  $n$  vertices and  $\mathcal{P} = \{P_i \mid i \in I\}$  is a collection of translates of  $P$  which forms an  $M$ -fold covering of the plane. We will set the value of  $M$  later. Let  $m$  be the minimum distance between any vertex and non-adjacent side of  $P$ . Take a square grid  $\mathcal{G}$  of basic distance  $m/2$ . Obviously, any translate of  $P$  intersects at most  $K = 4\pi(\text{diam}(P) + m)^2/m^2$  basic squares. For each (closed) basic square  $B$ , using its compactness, we can find a *finite* subcollection of the translates such that they still form an  $M$ -fold covering of  $B$ . Take the union of all these subcollections. We have a *locally finite*  $M$ -fold covering of the plane. That is, every compact set is intersected by finitely many of the translates. It is sufficient to decompose this covering. For simplicity, use the same notation  $\mathcal{P} = \{P_i \mid i \in I\}$  for this subcollection.

We formulate and solve the problem in its dual form. Let  $O_i$  be the center of gravity of  $P_i$ . Since  $\mathcal{P}$  is an  $M$ -fold covering of the plane, every translate of  $\bar{P}$ , the reflection of  $P$  through the origin, contains at least  $M$  points of the locally finite set  $\mathcal{O} = \{O_i \mid i \in I\}$ .

The collection  $\mathcal{P} = \{ P_i \mid i \in I \}$  can be decomposed into two coverings if and only if the set  $\mathcal{O} = \{ O_i \mid i \in I \}$  can be colored with two colors, such that every translate of  $\bar{P}$  contains a point of both colors.

Let  $\mathcal{W} = \{ W_1, W_2, \dots, W_n \}$  be the set of wedges that correspond to the vertices of  $\bar{P}$ . By the convexity of  $\bar{P}$ , no pair  $\{ W_i, W_j \}$  is of type 5 (Special), therefore, by the previous Lemmas, each pair is NC. Consequently, by Lemma 8,  $\mathcal{W}$  is NC as well. So there is a  $k$  with the following property.

\* Any set of points  $S$  can be colored with two colors such that any translate of  $W_1, W_2, \dots, W_n$  that contains at least  $k$  points of  $S$ , contains points of both colors.

Choose  $M$  such that  $M \geq kK$ , and color the points of  $\mathcal{O}$  in each basic square separately, with property \*.

Since any translate  $P'$  of  $\bar{P}$  intersects at most  $K$  basic squares of the grid  $\mathcal{G}$ ,  $P'$  contains at least  $M/K \geq k$  points of  $\mathcal{O}$  in the same basic square  $B'$ . By the choice of the grid  $\mathcal{G}$ ,  $B'$  contains at most one vertex of  $P'$ , hence  $B' \cap P' = B' \cap W$ , where  $W$  is a translate of some  $W_i \in \mathcal{W}$ . So, by property \*,  $P'$  contains points of  $\mathcal{O} \cap B'$  of both colors. This concludes the proof of Theorem 1.  $\square$

## 4 Concluding Remarks

Throughout this paper we made no attempt to optimize the constants. However, it may be an interesting problem to determine (asymptotically) the smallest  $k$  in the proof of Theorem 1.

Another interesting question is to decide whether this constant depends only on the number of vertices of the polygon or on the shape as well. In particular, we cannot verify the following.

**Conjecture.** *There is a constant  $k$  such that any  $k$ -fold covering of the plane with translates of a convex quadrilateral can be decomposed into two coverings.*

One can also investigate whether a given covering can be decomposed into  $s$  coverings, for some fixed  $s$ . For any planar set  $P$  and  $s \geq 1$ , if it exists, let  $k = k(s, P)$  be the smallest number such that any  $k$ -fold covering of the plane with translates of  $P$  can be decomposed into  $s$  coverings. By Theorem 1,  $k(2, P)$  exists for any open convex polygon  $P$ . With a slight modification of its proof, we get the following more general result.

**Theorem 1'.** *For any open convex polygon  $P$ , and any  $s$ , there exists a (smallest) number  $k = k(s, P)$  such that any  $k$ -fold covering of the plane with translates of  $P$  can be decomposed into  $s$  coverings.*

Our proof gives  $k(s, P) < K_P(8s)^{2^{n-1}}$ , where  $K_P$  is the constant  $K$  from the proof of Theorem 1 and  $n$  is the number of vertices of  $P$ . For *centrally symmetric* open convex polygons Pach and Tóth [PT07] established a much better upper bound, which is quadratic in  $s$ . Recently, Aloupis et al. [A08] improved it to linear. They showed that, for any centrally symmetric open convex polygon  $P$ , there is an  $\alpha_P$  such that  $k(s, P) < \alpha_P s$ . The best known lower bound on  $k(s, P)$  is  $(s, P) \geq \lfloor 4s/3 \rfloor - 1$  [PT07].

Our proofs use the assumption that the covering is locally finite, and for open polygons we could find a locally finite subcollection which is still a  $k$ -fold covering. Still, we strongly believe that Theorem

1 holds for closed convex polygons as well.

**Acknowledgement.** We are very grateful to the anonymous referee for his remarks.

## References

- [A08] G. Aloupis, J. Cardinal, S. Collette, S. Langerman, D. Orden, P. Ramos: Decomposition of Multiple Coverings into More Parts, in preparation, arXiv:0807.0552v1.
- [BMP05] P. Brass, J. Pach, and W. Moser: *Research Problems in Discrete Geometry*, Springer, Berlin, 2005.
- [MP86] P. Mani-Levitska, J. Pach: Decomposition problems for multiple coverings with unit balls, manuscript, 1986.
- [P80] J. Pach: Decomposition of multiple packing and covering, *Diskrete Geometrie*, 2. Kolloq. Math. Inst. Univ. Salzburg, 169-178, 1980.
- [P86] J. Pach: Covering the plane with convex polygons, *Discrete and Computational Geometry* **1** (1986), 73-81.
- [PTT05] J. Pach, G. Tardos, G. Tóth: Indecomposable coverings, In: The China–Japan Joint Conference on Discrete Geometry, Combinatorics and Graph Theory (CJCDGCGT 2005), Lecture Notes in Computer Science **4381**, Springer, Berlin, 2007.
- [PT07] J. Pach, G. Tóth: Decomposition of multiple coverings into many parts, in: *Proc. 23rd ACM Symposium on Computational Geometry, (SoCG07) 2007*, 133-137. Also in: *Comput. Geom. Theory and Applications*, accepted.
- [P08] D. Pálvölgyi: Indecomposable coverings with concave polygons, in preparation.
- [TT07] G. Tardos, G. Tóth: Multiple coverings of the plane with triangles, *Discrete and Computational Geometry* **38** (2007), 443-450.