Almost Cross-Intersecting and Almost Cross-Sperner Pairs of Families of Sets

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Abstract

For a set G and a family of sets $\mathcal{F}$ let $\mathcal{D}_{\mathcal{F}}(G) = \{F \in \mathcal{F} : F \cap G = \emptyset\}$ and $\mathcal{S}_{\mathcal{F}}(G) = \{F \in \mathcal{F} : F \subseteq G \text{ or } G \subseteq F\}$. We say that a family is l-almost intersecting, $(\leq l)$-almost intersecting, l-almost Sperner, $(\leq l)$-almost Sperner if $|\mathcal{D}_{\mathcal{F}}(F)| = l$, $|\mathcal{D}_{\mathcal{F}}(F)| \leq l$, $|\mathcal{S}_{\mathcal{F}}(F)| = l$, $|\mathcal{S}_{\mathcal{F}}(F)| \leq l$ (respectively) for all $F \in \mathcal{F}$. We consider the problem of finding the largest possible family for each of the above properties. We also address the analogous generalization of cross-intersecting and cross-Sperner families.

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1 Introduction

We use standard notation: $[n]$ denotes the set of the first n positive integers, 2^S denotes the power set of the set S and $\binom{S}{k}$ denotes the set of all k-element subsets of S. The complement of a set F is denoted by $\overline{F}$ and for a family $\mathcal{F}$ we write $\overline{\mathcal{F}} = \{\overline{F} : F \in \mathcal{F}\}$.

A family $\mathcal{F}$ of sets is called *intersecting* if any pair of sets in $\mathcal{F}$ have a non-empty intersection and is called an *antichain* or *Sperner family* if there is no pair of sets in $\mathcal{F}$ such that one set contains the other. Sperner's theorem [16] states that an antichain $\mathcal{F} \subseteq 2^{[n]}$ has size at most $\binom{n}{\lfloor n/2 \rfloor}$ and equality holds if and only if $\mathcal{F} = \binom{[n]}{\lfloor n/2 \rfloor}$ or $\mathcal{F} = \binom{[n]}{\lceil n/2 \rceil}$. A theorem of Erdős, Ko and Rado [4] states that an intersecting family $\mathcal{F} \subseteq \binom{[n]}{k}$ has size at most $\binom{n-1}{k-1}$ provided $2k \leq n$, furthermore, if $2k < n$, then equality holds if and only if $\mathcal{F}$ is the set of all k-subsets of $[n]$ containing a fixed element of $[n]$. These are the the two basic results in extremal finite set theory both of which have many generalizations and extensions.

One such generalization of intersecting families was introduced by the present authors in [8]. For a set G and a family $\mathcal{F}$ let us write $\mathcal{D}_{\mathcal{F}}(G) = \{F \in \mathcal{F} : F \cap G = \emptyset\}$. Then we say that a family $\mathcal{F}$ of sets is *l-almost intersecting* (*$(\leq l)$-almost intersecting*) if $|\mathcal{D}_{\mathcal{F}}(F)| = l$ ($|\mathcal{D}_{\mathcal{F}}(F)| \leq l$) holds for all $F \in \mathcal{F}$. An analogous generalization of Sperner families is the following: for a set G and a family $\mathcal{F}$ let us write $\mathcal{S}_{\mathcal{F}}(G) = \{F \in \mathcal{F} : F \subseteq G \text{ or } G \subseteq F\}$ and call a family $\mathcal{F}$ *l-almost Sperner* (*$(\leq l)$-almost Sperner*) if $|\mathcal{S}_{\mathcal{F}}(F)| = l$ ($|\mathcal{S}_{\mathcal{F}}(F)| \leq l$) holds for all $F \in \mathcal{F}$. Note that a 0-almost intersecting family is intersecting and a 1-almost Sperner family is Sperner.

In [8] we considered the problem of finding the largest possible l-almost intersecting and $(\leq l)$-almost intersecting families. In both cases we addressed the problem of uniform and non-uniform families. Among other things, we proved that if $\mathcal{F}$ is a k-uniform l-almost intersecting family, then $|\mathcal{F}| \leq C(k,l)$, where the constant is independent of the size of the ground set. We were only able to determine the best bound in the case if $k = 2$ or if $l = 1$. The latter is an immediate consequence of the following theorem of Bollobás [3].

Theorem 1.1 (Bollobás [3]). *If the pairs $(A_i, B_i)_{i=1}^m$ satisfy that $A_i \cap B_j = \emptyset$ if and only if $i = j$, then the following inequality holds:*

$$\sum_{i=1}^{m} \frac{1}{\binom{|A_i|+|B_i|}{|A_i|}} \leq 1,$$

in particular if $|A_i| \leq k$ and $|B_i| \leq r$ for all $1 \leq i \leq m$, then $m \leq \binom{k+r}{k}$ and equality holds if and only if the pairs are all possible partitions into sets of size k and r of some $(k+r)$-set X.

In [8], we made the following conjecture:

Conjecture 1.2. *For any k there exists $l_0 = l_0(k)$ such that if $l \geq l_0$ and $\mathcal{F}$ is a k-uniform l-almost intersecting family, then $|\mathcal{F}| \leq (l+1)\binom{2k-2}{k-1}$. If true, this is sharp as shown by the family $\mathcal{F}^* = \{F' \cup \{x\} : F' \in \binom{[2k-2]}{k-1}, x \in \{2k-1, 2k-2, ..., 2k+l-1\}\}$.*

In Section 2, we prove the following upper bound which is much closer to what Conjecture 1.2 states than what we proved in [8].

Theorem 1.3. *Let $\mathcal{F}$ be an l-almost intersecting family of k-sets. Then $|\mathcal{F}| \leq (2l - 1) \cdot \binom{2k}{k}$.*

We will prove Theorem 1.3 in a more general form. To do so we will consider the generalizations of cross-intersecting (see among others [2, 3, 5, 6, 7, 12, 13, 14]) and cross-Sperner (see [9]) pairs of families. We call the pair $(\mathcal{F}, \mathcal{G})$ of families *l-almost cross-intersecting* (*$(\leq l)$-almost cross-intersecting*) if $|\mathcal{D}_{\mathcal{F}}(G)| = l$ ($|\mathcal{D}_{\mathcal{F}}(G)| \leq l$) and $|\mathcal{D}_{\mathcal{G}}(F)| = l$ ($|\mathcal{D}_{\mathcal{G}}(F)| \leq l$) holds for all $F \in \mathcal{F}$ and $G \in \mathcal{G}$. Similarly, the pair $(\mathcal{F}, \mathcal{G})$ is *l-almost cross-Sperner* (*$(\leq l)$-almost cross-Sperner*) if $|\mathcal{S}_{\mathcal{F}}(G)| = l$ ($|\mathcal{S}_{\mathcal{F}}(G)| \leq l$) and $|\mathcal{S}_{\mathcal{G}}(F)| = l$ ($|\mathcal{S}_{\mathcal{G}}(F)| \leq l$) holds for all $F \in \mathcal{F}$ and $G \in \mathcal{G}$.

There are two standard ways to measure the size of a pair of families $(\mathcal{F}, \mathcal{G})$: either by the product $|\mathcal{F}| \cdot |\mathcal{G}|$ or by the sum $|\mathcal{F}| + |\mathcal{G}|$. Note that if l is at least 1 and we are considering l-almost cross-intersecting or l-almost cross-Sperner pairs, then immediately $|\mathcal{F}| = |\mathcal{G}|$ holds, thus it makes no difference whether we measure by the product or by the sum.

There is a very natural way to create pairs of families with the above properties: if $\mathcal{F}$ is, e.g., l-almost intersecting, then the pair $(\mathcal{F}, \mathcal{F})$ is l-almost cross-intersecting. Clearly, the other three analogous statements hold.

Problem 1.4. *Under what conditions is it true that for a maximum size family $\mathcal{F} \subseteq 2^{[n]}$ with one of the above properties, the pair $(\mathcal{F}, \mathcal{F})$ maximizes $|\mathcal{F}| \cdot |\mathcal{G}|$ or $|\mathcal{F}| + |\mathcal{G}|$ over the pairs $(\mathcal{F}, \mathcal{G})$ with the corresponding "cross-property"?*

One can easily answer the above problem in the negative for $(\leq l)$-almost cross-Sperner pairs. Indeed, for any l an $(\leq l)$-almost Sperner family cannot contain an $(l+1)$-*fork*, i.e. sets $F_0, F_1, ..., F_{l+1}$ with $F_0 \subset F_i$ for all $1 \leq i \leq l+1$. A result of Katona and Tarján [10] states that such a family can have size at most $(1+o(1))\binom{n}{\lfloor n/2 \rfloor}$, while the pair $\mathcal{F}^* = \{F \subset [n] : 1 \in F, 2 \notin F\}, \mathcal{G}^* = \{G \subset [n] : 1 \notin G, 2 \in G\}$ shows that the extremal pair cannot be a duplication of an extremal $(\leq l)$-almost Sperner family.

The rest of Section 2 contains some results which answer some instances of Problem 1.4 for almost cross-intersecting pairs of families. In particular, we will prove the following more general version of Theorem 1.3.

Theorem 1.5. *Let $(\mathcal{F},\mathcal{G})$ be a pair of l-almost intersecting families where $\mathcal{F}$ is r-uniform and $\mathcal{G}$ is k-uniform. Then $|\mathcal{F}|=|\mathcal{G}|\le(2l-1)\cdot\binom{k+r}{k}$.*

In Section 3, we prove the following theorem on 2-almost Sperner and (≤ 2)-almost Sperner families. The maximum size of such families was first determined by Katona and Tarján [10] and then reproved by Katona [11] in a different context. We use Katona's proof to determine all extremal families.

Theorem 1.6. *If $\mathcal{F}\subseteq 2^{[n]}$ is 2-almost Sperner or (≤ 2)-almost Sperner, then $|\mathcal{F}|\le 2\binom{n-1}{\lfloor n/2\rfloor}$. The only families with this size are isomorphic to $\mathcal{F}_1=\binom{[n-1]}{\lfloor n/2\rfloor}\cup\{F\cup\{n\}: F\in\binom{[n-1]}{\lfloor n/2\rfloor}\}$, $\mathcal{F}_2=\overline{\mathcal{F}_1}$ and, if n is even, $\mathcal{F}_3=\binom{[n]}{n/2}$.*

2 Almost intersecting families

We start this section by proving Theorem 1.5.

Proof. Let $(\mathcal{F},\mathcal{G})$ be as stated in the theorem. We define the graph Γ in the following way: let $V(\Gamma)=\{(F,G): F\in\mathcal{F}, G\in\mathcal{G}, F\cap G=\emptyset\}$ and $E(\Gamma)=\{((F,G)(F',G')): F=F'$ or $G=G'$ but not both $\}$. As the pair $(\mathcal{F},\mathcal{G})$ is l-almost intersecting, the maximum degree of Γ is at most $2(l-1)$, thus Γ can be properly colored with $2l-1$ colors. For any color class C of any proper coloring, consider the set of pairs (F,G) the corresponding vertices of which belong to C. Clearly, these pairs satisfy the conditions of Theorem 1.1, thus their number is at most $\binom{k+r}{k}$. Now the theorem follows by summing these bounds for all color classes. $\square$

Note that if we let both k and l tend to infinity, then the upper bound of Conjecture 1.2 is of the same order of magintude as the proved upper bound of Theorem 1.1. More precisely, asypmtotically they differ by a factor of eight.

We continue with non-uniform l-almost cross-intersecting pairs of families. Concerning Problem 1.4 let us make the following two easy observations. If $(\mathcal{F},\mathcal{G})$ is a 1-almost cross-intersecting pair, then just as in the proof of Theorem 1.5 sets of $\mathcal{F}$ and $\mathcal{G}$ can be partitioned into pairs that satisfy the condition of Theorem 1.1 and thus $|\mathcal{F}|=|\mathcal{G}|\le\binom{n}{\lfloor n/2\rfloor}$. This bound is tight as shown by the pair $\mathcal{F}=\binom{[n]}{\lfloor n/2\rfloor}$, $\mathcal{G}=\binom{[n]}{\lceil n/2\rceil}$. On the other hand, in [8] we showed that a 1-almost intersecting family cannot contain more than $2\binom{n-1}{\lfloor n/2\rfloor-1}$ sets if n is odd which is strictly smaller than $\binom{n}{\lfloor n/2\rfloor}$. Thus in this case the answer to Problem 1.4 is negative.

In [8], we also proved that the maximum size of a 2-almost intersecting family is $2\binom{n-2}{(n-2)/2}$ if n is even and $4\binom{n-3}{\lfloor n/2\rfloor-2}$ if n is odd. The proof involves several steps and some calculation each of which remains valid for 2-almost cross-intersecting pairs of families. Thus, the answer to Problem 1.4 is positive for $l = 2$ as opposed to the case $l = 1$.

Let us finish this section by listing some easy results about ($\leq l$)-almost cross-intersecting pairs of families. As all proofs are either trivial or simple alterations of proofs from [8], we omit most details. In the case where $\mathcal{F}$ and $\mathcal{G}$ form an ($\leq l$)-almost cross-intersecting pair families where $\mathcal{F}$ is r-uniform and $\mathcal{G}$ is k-uniform we will assume that $\binom{n}{r} \geq \binom{n}{k}$. It it is easy to see that if $\mathcal{F}$ and $\mathcal{G}$ are as above, then $|\mathcal{F}| \cdot |\mathcal{G}| \leq \binom{n-1}{k-1}\binom{n-1}{r-1}$ and $|\mathcal{F}| + |\mathcal{G}| \leq \binom{n}{r}$ provided n is large enough. If, in addition, we require both $\mathcal{F}$ and $\mathcal{G}$ to be non-empty, then one can obtain $|\mathcal{F}| + |\mathcal{G}| \leq l + 1 + \binom{n}{r} - \binom{n-k}{r}$. All these bounds are sharp as seen by the pairs $\mathcal{F}_1 = \{F \in \binom{[n]}{r} : 1 \in F\}, \mathcal{G}_1 = \{G \in \binom{[n]}{k} : 1 \in G\}$; $\mathcal{F}_2 = \binom{[n]}{r}, \mathcal{G}_2 = \emptyset$ and $\mathcal{G}_3 = \{G\}$, $\mathcal{F}_3 = \{F \in \binom{[n]}{r} : F \cap G \neq \emptyset\} \cup \{H_1, H_2, ..., H_l\}$ where G is any k-subset of $[n]$ and all H_i are disjoint from G.

Let us consider Problem 1.4 for pairs ($\mathcal{F}$,$\mathcal{G}$) of non-uniform families. In [8], we determined the maximum size of an ($\leq l$)-almost intersecting family for $l = 1, 2$. In that case when n is even the proofs from [8] can be used to show that $|\mathcal{F}| \cdot |\mathcal{G}| \leq M^2$ and $|\mathcal{F}| + |\mathcal{G}| \leq 2M$ where M is the maximum size of an ($\leq l$)-almost intersecting family i.e. $M = \sum_{k=n/2}^{n} \binom{n}{k}$. If n is odd, then Problem 1.4 for $l = 1$ can be settled in the negative as $|\mathcal{F}_0| + |\mathcal{G}_0| > 2|\mathcal{F}_1|$ where $\mathcal{F}_0 = \{F \subseteq [n] : |F| > n/2\}, \mathcal{G}_0 = \mathcal{F}_0 \cup \binom{[n]}{\frac{n-1}{2}}$ and $\mathcal{F}_1 = \mathcal{F}_0 \cup \{F \in \binom{[n]}{\frac{n-1}{2}} : 1 \in F\}$. Note that $\mathcal{F}_1$ was shown to be largest (≤ 1)-almost intersecting family in [8] and altering that proof a little, one can see that $|\mathcal{F}_0| + |\mathcal{G}_0|$ maximizes the sum over all pairs. If $l = 2$, then the answer to Problem 1.4 is positive. As before the proof is almost identical to the one in [8].

3 Almost Sperner families

We start this section by proving Theorem 1.6. As we mentioned in Section 1, the bound stated in the theorem was first proved by Katona and Tarján [10] and recently reproved by Katona [11]. We will sketch Katona's proof (but omit some tedious calculations) as our argument will use it heavily to determine the extremal families. Before starting the proof let us describe the context of both previous papers and introduce some terminology. The *comparability graph* $G(P)$ of a poset P is a directed graph with vertex set P and (p, q) is an arc if and only if $p \prec_P q$. The *components* of a family $\mathcal{F} \subseteq 2^{[n]}$ are the subfamilies of which the corresponding vertices form an undirected component of $G(P_{\mathcal{F}})$, where $P_{\mathcal{F}}$ is the subposet of the Boolean poset induced by $\mathcal{F}$. We say that the poset P contains another poset Q if $G(Q)$ is a (not necessarily induced) subgraph of $G(P)$ in the directed sense. For any set $\mathcal{P}$ of posets

$La(n, \mathcal{P})$ denotes the maximum size that a family $\mathcal{F} \subseteq 2^{[n]}$ can have such that $P_{\mathcal{F}}$ does not contain any $P \in \mathcal{P}$.

Let $\bigvee$ denote the 3-element poset with a smallest element and two other non-comparable element and let $\bigwedge$ denote the 3-element poset obtained from $\bigvee$ by reversing the orientation of all arcs in $G(\bigvee)$. Clearly, a family $\mathcal{F} \subseteq 2^{[n]}$ is (≤ 2)-almost Sperner if and only if the subposet of the Boolean poset B_n induced by $\mathcal{F}$ contains neither $\bigvee$ nor $\bigwedge$. Now we are ready to prove Theorem 1.6.

Proof. The 2-almost Sperner ((≤ 2)-almost Sperner) property implies that the connected components of $\mathcal{F}$ are either isolated points or pairs of sets. Let α_1 and α_2 denote their respective numbers. Then $|\mathcal{F}| = \alpha_1 + 2\alpha_2$. Let $c(P(1,a))$ ($c(P(2,a_1,a_2))$) denote the number of full chains going through a one element component (two element component) where the size of the set in the component is a (a_1, a_2). Katona [11] showed that

$$c(P(1,a)) \geq \left\lfloor \frac{n}{2} \right\rfloor ! \left\lceil \frac{n}{2} \right\rceil ! \qquad\qquad c(P(2,a_1,a_2)) \geq n \left\lfloor \frac{n-1}{2} \right\rfloor ! \left\lceil \frac{n-1}{2} \right\rceil !, \tag{1}$$

where equality holds if and only if $a = \lfloor \frac{n}{2} \rfloor$ or $a = \lceil \frac{n}{2} \rceil$ and $a_1 = a_2 - 1 = \lfloor \frac{n-1}{2} \rfloor$ or $a_1 = a_2 - 1 = \lceil \frac{n-1}{2} \rceil$. Let us count the pairs $(C, \mathcal{C})$ where C is a connected component of $\mathcal{F}$ and $\mathcal{C}$ is a full chain in $[n]$. As any full chain can meet at most one component, by (1) we obtain

$$\alpha_1 \left\lfloor \frac{n}{2} \right\rfloor ! \left\lceil \frac{n}{2} \right\rceil ! + 2\alpha_2 \frac{n}{2} \left\lfloor \frac{n-1}{2} \right\rfloor ! \left\lceil \frac{n-1}{2} \right\rceil ! \leq n!. \tag{2}$$

As $\frac{n}{2} \lfloor \frac{n-1}{2} \rfloor ! \lceil \frac{n-1}{2} \rceil ! \leq \lfloor \frac{n}{2} \rfloor ! \lceil \frac{n}{2} \rceil !$ with equality if and only if n is even, we obtain

$$(\alpha_1 + 2\alpha_2) \frac{n}{2} \left\lfloor \frac{n-1}{2} \right\rfloor ! \left\lceil \frac{n-1}{2} \right\rceil ! \leq n!. \tag{3}$$

Rearranging gives the bound on $|\mathcal{F}| = \alpha_1 + 2\alpha_2$.

Note that to prove that a particular 2-almost Sperner (or (≤ 2)-almost Sperner) family is not maximal it is enough to show that there is at least one full chain that does not meet $\mathcal{F}$ and thus strict inequality holds in (3). Also, by the above computations, we know that if $\mathcal{F}$ is of maximum size, then the connected components of $\mathcal{F}$ must be of the following types:

- if n is even, isolated points with corresponding sets of size $\frac{n}{2}$, pairs of sets with sizes $\frac{n}{2} - 1$ and $\frac{n}{2}$ or $\frac{n}{2}$ and $\frac{n}{2} + 1$,
- if n is odd, pairs of sets with sizes $\lfloor \frac{n}{2} \rfloor$ and $\lceil \frac{n}{2} \rceil$.

We start with two claims that reduce the problem in the case when n is even to the situation when $\mathcal{F}$ contains only one type of components.

Claim 3.1. *If n is even and $\mathcal{F}$ contains both isolated sets and pairs of sets as connected components, then $\mathcal{F}$ is not of maximum size.*

Proof. Let F be an isolated set of $\mathcal{F}$ and $H_1 \subsetneq H_2$ be a pair of sets in $\mathcal{F}$ such that $d(F, H_1) + d(F, H_2)$ is minimal (where d denotes the Hamming distance of F and H_i). We know that $|F| = \frac{n}{2}$ and, by changing to $\overline{\mathcal{F}}$ if necessary, we may assume that $|H_1| = |H_2| - 1 = \frac{n}{2}$. First let us assume that $d(F, H_1) + d(F, H_2) = 5$, which is the minimum possible. Let $C = F \cap H_1$, $\{x\} = F \setminus H_2$ and $\{z\} = H_2 \setminus H_1$ and consider any full chain $\mathcal{C}$ containing $A = C \cup \{z\}$ and $B = C \cup \{x, z\}$. By the assumption on the sizes of sets in $\mathcal{F}$, we know that $\mathcal{F} \cap \mathcal{C} \subseteq \{A, B\}$, but A is a proper subset of H_2 different from H_1, thus $A \notin \mathcal{F}$ and B is a proper superset of F, thus $B \notin \mathcal{F}$. We proved that $\mathcal{F} \cap \mathcal{C} = \emptyset$ which shows that (3) cannot hold with equality and thus $\mathcal{F}$ cannot be of maximum size.

Let us now assume that $d(F, H_1) + d(F, H_2) > 5$. We consider two cases: if $H_2 \setminus H_1 \not\subseteq F$, then fix an arbitrary $f \in F \setminus H_1$ and write $h = H_2 \setminus H_1$. Consider a full chain containing $G_1 = (F \setminus \{f\}) \cup \{h\}$ and $G_2 = F \cup \{h\}$. Again, by the assumption on the sizes of sets in $\mathcal{F}$, we know that $\mathcal{F} \cap \mathcal{C} \subseteq \{G_1, G_2\}$, but G_2 is a proper superset of F, thus $G_2 \notin \mathcal{F}$, while G_1 is closer to the pair (H_1, H_2) than F and also if G_1 was contained in a pair component of $\mathcal{F}$, then its "distance" from F would be 5, thus $G_1 \notin \mathcal{F}$. We showed the existence of a chain $\mathcal{C}$ so that $\mathcal{F} \cap \mathcal{C} = \emptyset$, thus (3) cannot hold with equality and thus $\mathcal{F}$ cannot be of maximum size. □

Claim 3.2. *If n is even and $\mathcal{F}$ contains pairs of sets as connected components with different set sizes, then $\mathcal{F}$ is not of maximum size.*

Proof. Suppose not and let us pick two components $F_1 \subsetneq F_2$ and $G_1 \subsetneq G_2$ of $\mathcal{F}$ such that the sizes of the sets are different (and thus, by the optimality of $\mathcal{F}$, $\frac{n}{2} = |F_1| + 1 = |F_2| = |G_1| = |G_2| - 1$) and $d(F_2, G_1)$ is minimal. Let us first assume that $d(F_2, G_1) = 2$ and let x be the single element of $F_2 \setminus F_1$ and let u be the single element of $F_2 \setminus G_1$. As $d(F_2, G_1) = 2$ and $F_1 \not\subseteq G_1$ we obtain $x \neq u$ and $x \in G_1$. Consider any full chain $\mathcal{C}$ going through $G_1 \setminus \{x\}$ and $(G_1 \setminus \{x\}) \cup \{u\}$. $\mathcal{C}$ cannot contain any set from $\mathcal{F}$ as its members with size at most $\frac{n}{2} - 1$ are proper subsets of G_1 and those with size at least $\frac{n}{2}$ are proper supersets of F_1 different from F_2, therefore these chains show that (3) cannot hold with equality, which contradicts $\mathcal{F}$ being of maximum size.

Now suppose $d(F_2, G_2) > 2$ and let us fix $x \in F_1 \setminus G_2$ and $y \in G_1 \setminus F_2$. Consider any full chain $\mathcal{C}$ going through $A = F_2 \setminus \{x\}, B = A \cup \{y\}, C = B \cup \{x\} = F_2 \cup \{y\}$. By the maximality of $\mathcal{F}$ we have $\mathcal{C} \cap \mathcal{F} \subseteq \{A, B, C\}$, but A is a subset of F_2 different from F_1, C is a superset of F_2 and thus A and C cannot be in $\mathcal{F}$. While $B \in \mathcal{F}$ would contradict the minimality of $d(F_2, G_2)$. Again the existence of such $\mathcal{C}$ shows that (3) cannot hold with equality, which contradicts $\mathcal{F}$ being of maximum size. $\square$

Claim 3.3. *If all components of $\mathcal{F}$ are pairs of the same type but for two pairs $F_1 \subsetneq F_2$ and $H_1 \subsetneq H_2$ of sets in $\mathcal{F}$ we have $F_2 \setminus F_1 \neq H_2 \setminus H_1$, then $\mathcal{F}$ is not of maximum size.*

Proof. Again, pick the two pairs such that the sum $d(H_1, F_1)+d(H_1, F_2)+d(H_2, F_1)+d(H_2, F_2)$ is minimal. This sum is at least 12 by the assumption $F_2 \setminus F_1 \neq H_2 \setminus H_1$. Let us first assume that this sum is 12. Then let h be the only element of $H_2 \setminus H_1$, f be the only element of $F_1 \setminus H_1$ and let $C = F_1 \cap H_1$. Consider any full chain $\mathcal{C}$ containing $A = C \cup \{h\}$ and $B = C \cup \{h, f\}$. By the assumption on the sizes of sets in $\mathcal{F}$, we know that $\mathcal{F} \cap \mathcal{C} \subseteq \{A, B\}$, but A is a proper subset of H_2 different from H_1, thus $A \notin \mathcal{F}$ and B is a proper superset of F_1 different from F_2, thus $B \notin \mathcal{F}$. We proved that $\mathcal{F} \cap \mathcal{C} = \emptyset$ which shows that (3) cannot hold with equality and thus $\mathcal{F}$ cannot be of maximum size.

Let us now assume that the sum mentioned above is strictly larger than 12. Let h be an element of $H_1 \setminus F_1$ and if the only element x of $F_2 \setminus F_1$ is contained in H_1, then let $h = x$. Furthermore, let f be an element of $F_1 \setminus H_2$. Consider a full chain $\mathcal{C}$ containing $G_1 = H_2 \setminus \{h\}$ and $G_2 = (H_2 \setminus \{h\}) \cup \{f\}$. Again, by the assumption on the sizes of sets in $\mathcal{F}$, we have $\mathcal{F} \cap \mathcal{C} \subseteq \{G_1, G_2\}$. We have $H_1 \neq G_1 \subsetneq H_2$, thus $G_1 \notin \mathcal{F}$. Finally, observe that $G' \notin \mathcal{F}$ as the component (G', G_2) containing G_2 would be closer to the pair F_1, F_2 and as $x \notin G_2$ we would still have that $F_2 \setminus F_1 \neq G_2 \setminus G'$. $\square$

Claim 3.3 completes the proof of Theorem 1.6. $\square$

Now we turn our attention to Problem 1.4 in the case of almost cross-Sperner pairs of families.

Theorem 3.4. *If $(\mathcal{F}, \mathcal{G})$ is a 1-almost cross-Sperner pair of families with $\mathcal{F}, \mathcal{G} \subseteq 2^{[n]}$, then we have*

$$|\mathcal{F}| = |\mathcal{G}| \leq \binom{n}{\lfloor n/2 \rfloor}$$

and equality holds if and only if $\mathcal{F} = \mathcal{G} = \binom{[n]}{\lfloor n/2 \rfloor}$ or $\mathcal{F} = \mathcal{G} = \binom{[n]}{\lceil n/2 \rceil}$.

Proof. Let $(\mathcal{F}, \mathcal{G})$ be a 1-almost cross-Sperner pair of families and define $\mathcal{F}^d = \{F \in \mathcal{F} : \exists F' \in \mathcal{F}$ such that $F \subsetneq F'\}, \mathcal{F}^u = \{F \in \mathcal{F} : \exists F' \in \mathcal{F}$ such that $F \supsetneq F'\}, \mathcal{F}^{sp} = \mathcal{F} \setminus (\mathcal{F}^d \cup \mathcal{F}^u)$. First note that $\mathcal{F}^u \cap \mathcal{F}^d = \emptyset$. Indeed, otherwise there would exist three sets $F_1 \subsetneq F_2 \subsetneq F_3$ in $\mathcal{F}$ and the set $G \in \mathcal{G}$ in containment with F_2 would have two such sets from $\mathcal{F}$ contradicting the 1-almost cross-Sperner property of $(\mathcal{F}, \mathcal{G})$.

Let us define $\mathcal{G}' = \{G \in \mathcal{G} : \exists F \in \mathcal{F}^d \text{ with } F \subseteq G \text{ or } G \subseteq F\}$. By the 1-almost cross-Sperner property of $(\mathcal{F}, \mathcal{G})$ we know that $|\mathcal{F}^d| = |\mathcal{G}'|$ and thus writing $\mathcal{F}' = \mathcal{F}^u \cup \mathcal{F}^{sp} \cup \mathcal{G}'$ we have $|\mathcal{F}| = |\mathcal{F}'|$.

Claim 3.5. *The family $\mathcal{F}'$ is an antichain.*

Proof. By definition and the observation that $\mathcal{F}^u \cap \mathcal{F}^d = \emptyset$, there is no pair of sets in $\mathcal{F}^u \cup \mathcal{F}^{sp}$ containing one another. Furthermore, as all sets in $\mathcal{G}'$ have their containment pairs in $\mathcal{F}^d$, there is no pair of sets (F, G) containing one another with $F \in \mathcal{F}^u \cup \mathcal{F}^{sp}, G \in \mathcal{G}'$.

Suppose there exist $G_1 \subsetneq G_2$ with $G_1, G_2 \in \mathcal{G}'$ and consider their containment pairs $F_1, F_2 \in \mathcal{F}^d$. Note that $F_i \subset G_i$ as otherwise G_i would have two containment pairs in $\mathcal{F}$. But then $F_1, F_2 \subset G_2$ which contradicts the 1-almost cross-Sperner property of $(\mathcal{F}, \mathcal{G})$. This contradiction proves the claim. $\square$

Claim 3.5 and Sperner's theorem proves the bound on the size of $\mathcal{F}$ and $\mathcal{G}$. To see the uniqueness of the extremal families note that by the uniqueness part of Sperner's theorem we have $\mathcal{F}' = \binom{[n]}{\lfloor n/2 \rfloor}$ or $\mathcal{F}' = \binom{[n]}{\lceil n/2 \rceil}$. Clearly, Claim 3.5 remains true if we replace $\mathcal{F}'$ by $\mathcal{F}'' := \mathcal{F}^{sp} \cup \mathcal{F}^d \cup \mathcal{G}''$ where $\mathcal{G}'' = \{G \in \mathcal{G} : \exists F \in \mathcal{F}^u \text{ with } F \subseteq G \text{ or } G \subseteq F\}$. Thus we are done if n is even or if $\mathcal{F}^{sp} \neq \emptyset$.

The only remaining possibility that the statement of the theorem does not hold is if n is odd and $\mathcal{F}' = \mathcal{F}^u \cup \mathcal{G}' = \binom{[n]}{\lceil n/2 \rceil}$ and $\mathcal{F}'' = \mathcal{F}^d \cup \mathcal{G}'' = \binom{[n]}{\lfloor n/2 \rfloor}$. Consider the regular bipartite graph B with partite sets $\binom{[n]}{\lfloor n/2 \rfloor}$ and $\binom{[n]}{\lceil n/2 \rceil}$ where two sets are connected if one of them contains the other. A situation as above would partition the graph B into two induced matchings $B(\mathcal{F}^d, \mathcal{G}')$ and $B(\mathcal{G}'', \mathcal{F}^u)$. This is impossible as that would mean that if the vertex corresponding to a set F belongs to one of the matchings, then all sets F' with $d(F, F') = 2$ (where d denotes the Hamming distance of sets or equivalently the graph distance in B) belong to the other matching, and if $n \geq 3$, then we cannot place the following 3 sets in a way that possesses the above property: $F, F \setminus \{x\} \cup \{y\}, F \setminus \{z\} \cup \{y\}$ with $x, z \in F, y \notin F$. $\square$

An alternate proof to Theorem 3.4 is to count the number of pairs $(H, \mathcal{C})$, where H is a set from $\mathcal{F} \cup \mathcal{G}$ (here we consider $\mathcal{F} \cup \mathcal{G}$ as a multifamily, i.e. sets may appear twice if they belong to both $\mathcal{F}$ and $\mathcal{G}$) and $\mathcal{C}$ is a full chain. Any full chain contains at most 2 sets from $\mathcal{F} \cup \mathcal{G}$ as two sets from $\mathcal{F}$ and one from $\mathcal{G}$ (or the other way round) would contradict the 1-almost cross-Sperner property, while 3 sets from $\mathcal{F}$ (or from $\mathcal{G}$) is impossible by the argument above for $\mathcal{F}^u \cap \mathcal{F}^d = \emptyset$. Thus we obtain the following LYM-type inequality:

$$\sum_{F \in \mathcal{F}} \frac{1}{\binom{n}{|F|}} + \sum_{G \in \mathcal{G}} \frac{1}{\binom{n}{|G|}} \leq 2, \tag{4}$$

which immediately proves the bound on $|\mathcal{F}| = |\mathcal{G}|$ and the uniqueness of the extremal pair if n is even. If n is odd, then by (4) all sets in $\mathcal{F} \cup \mathcal{G}$ have size $\lceil n/2 \rceil$ or $\lfloor n/2 \rfloor$. There is nothing to prove if all sets belong to both $\mathcal{F}$ and $\mathcal{G}$. Let $F \in \mathcal{F} \setminus \mathcal{G}$ with $|F| = \lceil n/2 \rceil$. Then all subsets of F of size $\lfloor n/2 \rfloor$ belong to exactly one of $\mathcal{F}$ or $\mathcal{G}$. Indeed, all such sets H are in $\mathcal{F} \cup \mathcal{G}$ as otherwise a chain going through H and F would contain only one set from $\mathcal{F} \cup \mathcal{G}$ and thus (4) would hold with strict inequality. Also, if H belonged to both $\mathcal{F}$ and $\mathcal{G}$, that would contradict the 1-almost cross-Sperner property of the pair $(\mathcal{F}, \mathcal{G})$. Applying the same argument to the supersets of the subsets of F and to subsets of those and so forth we obtain that $\mathcal{F} \cup \mathcal{G} = \binom{[n]}{\lfloor n/2 \rfloor} \cup \binom{[n]}{\lceil n/2 \rceil}$ and all sets appear in exactly one of the families. Now the uniqueness follows just as in the previous proof.

Let us finish this section with a short remark concerning Problem 1.4 for $(\leq l)$-almost cross-Sperner pairs of families $(\mathcal{F}, \mathcal{G})$. Using the argument from [9] one can prove that if both $\mathcal{F}$ and $\mathcal{G}$ are non-empty, then $|\mathcal{F}| + |\mathcal{G}| \leq 3 + l + 2^n - 2^{\lfloor n/2 \rfloor} - 2^{\lceil n/2 \rceil}$ provided n is large enough.

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