Online and quasi-online colorings of wedges and intervals

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Abstract

We consider proper online colorings of hypergraphs defined by geometric regions. Among others we prove that there is an online coloring method that colors N intervals on the line using $\Theta(\log N/k)$ colors such that for every point p, contained in at least k intervals, there exist two intervals containing p and having different colors. We prove the corresponding (but not equivalent) result about online coloring quadrants in the plane that are parallel to a given fixed quadrant. These results contrast to recent results of the first and third author showing that in the quasi-online setting 12 colors are enough to color quadrants (independent of N and k). We also consider coloring intervals in the quasi-online setting. In all cases we present efficient coloring algorithms as well.

1 Introduction

The study of proper colorings of graphs and hypergraphs encompasses a vast area of research. Here we focus on online and quasi-online colorings of hypergraphs defined by certain geometric regions in the plane. Considered individually, both (quasi-)online colorings and colorings of geometric hypergraphs are well-studied problems. However their common intersection has been somewhat neglected; we attempt here to right this oversight.

The study of proper colorings of geometric hypergraphs has attracted attention not just because this is a very basic and natural theoretical problem but also because it has strong connections with conflict-free colorings (for a survey covering many of these aspects see [22]) which have applications in frequency assignment problems and other practical areas. This has inspired research in several directions, from cover-decomposability problems to conflict-free colorings. Regions for which proper colorings of the corresponding hypergraphs have been studied, include half-planes, [14, 13], discs [7, 23] and axis-parallel rectangles [4, 20, 1].

The most basic problem concerning online colorings of general graphs is to determine how many colors are needed to color an arbitrary graph. Gyárfás and Lehel [8] showed that no online algorithm can color all trees by a bounded number of colors. The performance ratio of an online-coloring algorithm is the maximum ratio of the

*Research supported by OTKA, grant NK 78439 and by OTKA under EUROGIGA project GraDR 10-EuroGIGA-OP-003.

†Research supported by OTKA, grant PD 83586, by OTKA under EUROGIGA project GraDR 10-EuroGIGA-OP-003 and the János Bolyai Research Scholarship of the Hungarian Academy of Sciences.

number of colors used to the chromatic number over all input graphs on n vertices. Lovász Saks, and Trotter [17] gave an algorithm that obtains a performance ratio of $O(n/\log^* n)$ from above, Halldórson and Szegedy showed that no algorithm can produce a better performace ration than $n/\log^2 n$. If the chromatic number, χ is fixed, Kierstead [15] showed sublinear upper bounds of $O(n^{1-1/\chi!})$ for the needed number of colors for online coloring.

Online coloring problems of hypergraphs have also been investigated, see for example the paper of Nagy and Imreh [10] or Halldórson [9]. Online colorings of geometric hypergraphs have also been studied. In particular the problem of online conflict-free coloring of points with respect to intervals is a well known open problem [5]; the exact magnitude of the needed number of colors has not yet been determined. Online proper colorings of geometric hypergraphs so far have gained less attention, though some problems have been explored. For example online coloring intersection graphs of unit discs was examined in [25].

In this paper we shall study this notion for coloring points with respect to translates of a fixed wedge and for coloring intervals on a line. These two problems have strong connections but are not equivalent. To illustrate the relation of these problems to the classical graph theoretic problems notice that whereas in the well-studied problem of online coloring interval graphs one has to color the intervals such that for a point p all intervals covering p have different colors, in our setting we just need a coloring where for a point covered by at least two (or k for some fixed k) colors not all intervals covering p have the same color. Section 2 contains the results about these online coloring problems.

Also, recently quasi-online colorings (objects come online and we have to maintain a valid coloring at any time, yet we know in advance all the forthcoming objects and the order in which they come) of objects has gained attention, as such problems can be used to solve corresponding offline higher dimensional problems. In [11] it was shown that coloring wedges in the plane in a quasi-online manner is equivalent to coloring wedges in the space offline. This motivated us to revisit a problem about quasi-online coloring intervals. Here we reprove earlier theorems about the needed number of colors. Our proofs are more transparent then the ones in [13] and also the algorithms we provide are more efficient as they need $O(n \log n)$ steps to color n intervals instead of the earlier algorithms that needed $O(n^2)$ steps. Section 3 contains the results about quasi-online coloring intervals.

Other variants of online colorings were regarded by Bar-Noy et al. [2] in the dual setting of coloring points on the line. The recent paper of Cardinal et al. [3] considers quasi-online rainbow-colorings in this dual setting. They prove that any finite family of points on the line can be colored quasi-online with c colors such that any interval that contains at least $f(c) = 8c - 5$ points contains points of all colors. Recently the corresponding result about quasi-online rainbow-colorings of wedges was proved by the first and the third author [12]. This implies a lower bound for quasi-online rainbow colorings of intervals as well, although the bound in [12] for $f(c)$ is double exponential in c. It is unkown if a linear bound will work for wedges. For more background see [12].

2 Online coloring wedges and intervals

2.1 Definitions

Here we first collect a few definition that will be useful throughout the text.

A *wedge* in the plain is the set of points $\{(x, y) \in \mathbb{R} \times \mathbb{R} | x \leq x_0 \ y \leq y_0\}$ for fixed x_0, y_0. A finite set of points S in the plain is *k-properly c-colored* if S is colored with c colors such that every wedge intersecting S in at least k points contains at least two points of different colors. A wedge containing points of only one color is said to be *monochromatic*. We will often refer to this as simply a coloring of the set of points; the parameters k and c will be obvious from the context.

An *online coloring* of a set of points is a (k-proper c-)coloring in which the points are revealed one at a time and must be colored as they are revealed. A *quasi-online coloring* of an ordered set of points $S = \{s_1, \ldots s_n\}$ is a k-proper c-coloring of S such that every subset $\{s_1, s_2, \ldots, s_i\}$, $i < n$ is also k-properly c-colored. In this case, the entire set S can be inspected before the coloring is produced.

2.2 Online coloring wedges

Theorem 1. *[11] Any finite family of wedges in the plane can be colored quasi-online with 2 colors such that any point contained by at least $k = 12$ of these wedges is not monochromatic.*

Gábor Tardos [24] asked whether such a coloring can be achieved in a completely online setting, possibly with a larger k and more colors (again such that all large wedges are non-monochromatic). It is easy to see that 2 colors are not enough to guarantee colorful wedges (i.e. there may be arbitrarily large monochromatic wedges) even when the points are restricted to a diagonal line. However, 3 colors (and $k = 2$) are enough if the points are restricted to a diagonal, see [11]. Here we prove that in general, for any c and k, there exists a method of placing points in the plane such that any online-coloring of these points with c colors will result in the creation of monochromatic wedges of size at least k.

Theorem 2. *There exists a method to give $N = 2^n - 1$ points in a sequence such that for any online-coloring method using c colors there will be c monochromatic wedges, $W_1, W_2, \ldots, W_c$, and nonnegative ingeters $x_1, \ldots, x_c$ such that for each i, the wedge W_i contains exactly x_i points colored with color i and $\sum x_i \geq n + 1$ if $n \geq 2$.*

Corollary 3. *No online-coloring method using c colors can avoid to make a monochromatic wedge of size $k + 1$ for some sequence of $N = 2^{ck} - 1$ points.*

For a collection of c-colored points in the plane, we define the associated color-vector to be a vector of length c where the i^{th} coordinate is the size of the largest wedge consisting only of points with color i. The size of the color-vector refers to the sum of its coordinates.

Proof of Theorem 2. By induction on the size of the color-vector. Clearly, one point gives a color-vector of size 1. Two points guarantee a color-vector of size 2 if they are placed diagonally from each other. Now we can place the third point diagonally between thefirst two if they had a different color or place it diagonally below them to get a color-vector of size 3 for three points.

By the inductive hypothesis, using at most $2^{n-1} - 1$ points, we can force a color-vector with size n. Moving southeast (i.e. so that all the new points are southeast from the previous ones) we repeat the procedure, again using at most $2^{n-1} - 1$ points we can force a second color-vector with sum n. If the two color-vectors are different then the whole point set has a color-vector of size at least $n + 1$. If they are the same then we put an additional point southwest from all the points of the first set of points but not interfering with the second set of points. Then as this point is colored with some color, i, the i^{th} coordinate of the first color-vector increases. (The rest of its coordinates will becomes 0.) Together with the wedges found in the second set of points, we can see that the size of the color-vector of the whole point set increased by one. Altogether we used at most $2(2^{n-1} - 1) + 1 = 2^n - 1$ points, as desired. □

What happens if c or k is fixed? The case when $c = 2$ was considered, eg., in [11]. It is not hard to see that using $2k - 1$ points, the size of the largest monochromatic wedge can be forced to be at least k and this is the best possible. For $c = 3$ a quadratic (in k) number of points is needed to force a monochromatic wedge of size k:

Proposition 4. *The following statements hold.*

1. *There exists a method of placing k^2 points such that any online 3-coloring of these points produces a monochromatic wedge of size at least k.*
2. *There exists a method of online 3-coloring $k^2 - 1$ points such that all monochromatic wedges have size less than k.*

We use the following terminology. Given a collection of 3-colored points in the plane, we say a new, uncolored point x is a *potential member* of a monochromatic wedge W, if by giving x the color of W, the size of W increases. Furthermore if x is a potential member of W, and giving x a color different from the color of W destroys the wedge W, then x *threatens* the wedge W.

Proof. To prove the first statement, consider the largest monochromatic wedge of each color after some points have already been placed and colored. Moving in the northwest / southeast directions label the wedges W_1, W_2 and W_3. It is clear that W_2 lies between the other two wedges. Note that it is possible to place a new point directly southwest of the points in W_2 such that the point is a potential member of all three wedges but only threatens W_2. Thus if the point is assigned the color of one of the other wedges (say W_1), the size of W_1 increments while W_3 remains the same and W_2 is destroyed (it is no longer monochromatic). Now suppose W_2 is not larger than either of the other two wedges. In this case, a point is placed, as described above, such that it is a potential member of all three wedges. If the point is assigned the color of W_1 or W_3 then W_2 is destroyed and while W_1 (or W_3) moves from size i to $i+1$, the $j \leq i$ points of W_2 are rendered ineffective for forming monochromatic wedges. On the other hand suppose W_2 has size larger than (at least) one of the other wedges (say W_3). Then we forget the wedge W_3 and proceed as above where there is a new wedge W_0 (of size 0) between W_1 and W_2. (We can think that in the previous step W_2 increased from i to $i + 1$ while the $j \leq i$ points of W_3 are destroyed.) As we proceed in this way, the sizes of the two "side" wedges only increase at each step while the "middle" wedge may be reduced to size 0 at some steps. However, the j points of the middle wedge are only destroyed when a side wedge increases from i to $i+1$ when $i \geq j$. Thus by destroying at most $2\binom{k}{2}$ points we can guarantee that the two side wedges have size $k-1$. Adding at most k more points to the middle, a monochromatic wedge of size k is guaranteed.

To prove the second statement we must assign colors to the points to avoid a monochromatic k-wedge. When a new point, x is given, consider those wedges of which x is a potential member. Note that at most two of these are not threatened by x. Let s be the size of the smallest wedge, W which is not threatened by x but of which x is a potential member. If by giving x the color of W at least s points are destroyed among the wedge(s) threatened by x, then give x this color. Otherwise give x the color different from the two non-threatened wedges. In this way we guarantee that a wedge only increases in size from i to $i+1$ if at the same time i other points are destroyed (i.e. rendered ineffective) or if two other wedges of size $i+1$ already exist. Therefore if only $2\sum_{i=1}^{k-2} i + 3(k-1) = k^2 - 1$ vertices are online-colored, we can avoid a monochromatic k-wedge. □

For $c \geq 4$ we can give an exponential (in ck) lower bound for the worst case:

Theorem 5. *For $c \geq 4$ we can online color with c colors any set of $N = O(1.22074^{ck})$ points such that throughout the process there is no monochromatic wedge of size k. Moreover, if c is large enough, then we can even online color $N = O(1.46557^{ck})$ points.*

Proof. Denote the colors by the numbers $\{1, \ldots, c\}$. A wedge refers to both an area in the plane as well as the collection of placed points which fall within that area. For brevity, we will often refer to maximal monochromatic wedges as simply wedges. If a wedge is not monochromatic, we will specifically note it. At each step, we define a partition of all the points which have come online in such a way that each set in the partition contains exactly one maximal monochromatic wedge. Two maximal monochromatic wedges are called *neighbors* if they are contained within a larger (non-monochromatic) wedge which contains no other monochromatic wedges. If the placement and coloring of a point cause a wedge to no longer be monochromatic, that wedge has been *killed.*

We now describe how to color a new point given that we have already colored some (or possibly no) points. If the new point is Northeast of an earlier point, it is given a different color from the earlier point. In this case no new wedges are created and no wedges increase in size. Otherwise the new point will eventually be part of a wedge. We want to make sure that the color of the point is distinct from its neighbors colors. In particular, consider the (at most) two wedges which are neighbors of wedges containing the point but which do not actually contain the point. From the c colors we disregard these two colors. From the remaining, we choose the color which first minimizes the size of the wedge containing the point and secondly minimizes the color (as a number from 1 to c.) This means that our order of preference is first to have size 1 wedge of color 1, then a size 1 wedge of color 2, ..., size 1 wedge of color c, size 2 wedge of color 1, ... etc. These rules determine our algorithm, now we have to see how effective it is.

To prove the theorem we show that the partition set associated with the newly created (or incremented) wedge is relatively large. Suppose this wedge is of color i and size j. Let $A_{i,j}$ denote the smallest possible size of the associated partition set. One can regard $A_{i,j}$ as the least number of points that are required to "build" a wedge W of size i of color j. For simplicity, we also use the notation $B_{c(i-1)+j} = A_{i,j}$. Note that this notation is well defined as j is always less than c. Thus we have $B_1 = B_2 = B_3 = 1$ and $B_4 = 2$.

It follows from our preferences that $B_i \le B_j$ if $i \le j$. Our goal is to give a good lower bound on $A_{k,1} = B_{c(k-1)+1}$.

Notice that when we create a new wedge, it will kill many points that were contained in previous wedges. More precisely, from our preferences we have $B_i \ge 1 + B_{i-3} + B_{i-4} + \ldots + B_{i-c}$ (where $B_i = 0$ for $i \le 0$). Note that B_{i-1} and B_{i-2} is missing from this sum because the coloring method must choose a color different from the new points two to-be-neighbors'.

From the solution of this recursion we know that the magnitude of B_i is at least q^i where q is the (unique, real, > 1) solution of $q^c = (q^{c-2} - 1)/(q - 1)$, which is equivalent to $q^{c+1} = q^c + q^{c-2} - 1$. Moreover, since trivially $B_i \ge 1 \ge q^{i-c}$ if $i \le c$, from the recursion we also have $B_i \ge q^{i-c}$ for all i. If we suppose $c \ge 4$, then $q \ge 1.22074$ and from this $B_{c(k-1)+1} \ge 2.22074^{k-2}$. As c tends to infinity, q tends (from below) to the real root of $q^3 = q^2 + 1$, which is ≥ 1.46557. From this we obtain that $B_{c(k-1)+1} \ge 1.46557^{c(k-2)}$ if c is large. Also, in the special case $k = 2$, we get the well known sequence A000930 (see [19]), which is at least 1.46557^c, if c is big enough. ☐

Summarizing, if we have $c \ge 4$ colors, the smallest N_0 number of points that forces a monochromatic wedge of size k is exponential in ck. Thus, if the number of colors, c, is given, these bounds give an estimate of $\Theta(\log N/c)$ on the size of the biggest monochromatic wedge in the worst case.

If we consider k fixed (and we want to use as few colors as possible), by the above bound the number of colors needed to avoid a monochromatic wedge of size k is $\Theta(\log N/k)$ for $k \ge 1$.

Corollary 6. *There is a method to color online N points in the plane using $\Theta(\log N/k)$ colors such that all monochromatic wedges have size strictly less than k.*

Recall that Theorem 2 stated that $N = 2^n - 1$ points can always force a size $n + 1$ color-vector (for the definition see the proof of Theorem 2). We remark that Theorem 5 implies a lower bound close to this bound too. Indeed, fix, eg., $c = 4$ and $k = \lceil n/4 \rceil$. If the number of points is at most $N = O(1.22074^n) = O(1.22074^{ck})$ then by Theorem 5 there is an online coloring such that at any time there is no monochromatic wedge of size k, thus the color-vector is always at most $4(k-1) < n$.

Suppose now that k is fixed and we want to use as few colors as possible without knowing in advance how many points will come, i.e. for k fix we want to minimize c without knowing N. To solve this, we alter our previous algorithm. (Note that we could also easily adjust the algorithm if for an unknown N we want to minimize $\min(c, k)$, or ck, the answer would be still logarithmic in N.) All this comes with the price of loosing a bit on the base of the exponent. The following theorem implies that for $k = 2$ (and thus also for any $k \ge 2$) we can color any set of $N = O(1.0905^{ck})$ points and if k is big enough then we can color any set of $N = O(1.1892^{ck})$ points.

Theorem 7. *For fixed $k \ge 1$ we can color a countable set of points such that for any c, and any $n < 2^{\lfloor (c+1)/4 \rfloor (k-1)}$, the first n points of the set are k-properly c-colored.*

Proof. We need to define a coloring algorithm and prove that it uses many colors only if there were many points. The coloring and the proof is similar to the proof of Theorem 5, we only need to change our preferences when coloring and because of this the analysis of the performance of the algorithm differs slightly too. We fix a c and an $N < 2^{\lfloor (c+1)/4 \rfloor (k-1)}$ for which we will prove the claim of the theorem (the coloring we define can obviously not depend on c or N, but it depends on k). Denote the colors by the numbers $\{1, 2, \ldots, c-1, c, \ldots\}$.

We can suppose again that every new point will be on the actual diagonal. When we add a point its color must still be different from its to-be-neighbors' and together with this point we still cannot have a monochromatic wedge of size k. Our primary preference now is that we want to keep $\lfloor c/4 \rfloor$ small where c is the color of the new point (as a number). Our secondary preference is that the size of the biggest wedge containing the new point should be small.

This means that our order of preference is first to have size 1 wedge of color 1, then a size 1 wedge of color 2, size 1 wedge of color 3, a size 1 wedge of color 4, a size 2 wedge of color 1, ..., a size $k-1$ wedge of color 1, size $k-1$ wedge of color 2, a size $k-1$ wedge of color 3, a size $k-1$ wedge of color 4, size 1 wedge of color 5, size 2 wedge of color 5, ... etc. These rules determine our algorithm, now we have to see how effective it is.

$A_{i,j}$ is defined as in the proof of Theorem 5. We only need to prove that $A_{i,j} \geq 2^{(k-1)(\lfloor j/4 \rfloor + i - 1)}$ as this means that if the algorithm uses the color $c+1$, then we had at least $A_{1,c+1} \geq 2^{(k-1)\lfloor (c+1)/4 \rfloor} > N$ points, a contradiction. Recall that $A_{i,j}$ denotes the least number of points that are required to "build" a wedge W of size i of color j.

We prove by induction. First, $A_{1,1} = A_{1,2} = A_{1,3} = A_{1,4} = 1$ indeed. By our preferences, whenever we introduce a size one wedge with color j, we had to kill at least two (four minus the two forbidden colors of the neighbors of the new point) points that have colors from the previous 4-tuple of colors and are contained in monochromatic wedges of size $k-1$. Thus $A_{1,j} \geq A_{k-1,4(\lfloor j/(4-1) \rfloor + 1)} + A_{k-1,4(\lfloor j/(4-1) \rfloor + 2} \geq 2 \cdot 2^{(k-1)(\lfloor j/(4-1) \rfloor + k-2)} = 2 \cdot 2^{(k-1)(\lfloor j/4 \rfloor - 1} = 2^{(k-1)(\lfloor j/4 \rfloor + 1 - 1}$. If we introduce a wedge of size $i > 1$ with color j, we had to kill at least two points that have colors from the same 4-tuple of colors as j and are contained in monochromatic wedges of size $i-1$. Thus in this case $A_{i,j} \geq A_{i-1,4(\lfloor j/4 \rfloor + 1} + A_{i-1,4(\lfloor j/4 \rfloor + 2} \geq 2 \cdot 2^{(k-1)(\lfloor j/4 \rfloor + i - 2)} = 2^{(k-1)(\lfloor j/4 \rfloor + i - 1}$. □

Proposition 8. *The online coloring methods guaranteed by the second part of Proposition 4, Theorem 5 and Theorem 7 run in $O(n \log n)$ time to color the first n points (even if we have a countable number of points and n is not known in advance).*

The proof of this proposition is omitted as it follows easily from the analysis of the algorithms.

2.3 Online coloring intervals

This section deals with the following *interval coloring problem*. Given a finite family of intervals on the real line, we want to online color them with c colors such that throughout the process if a point is covered by at least k intervals, then not all of these intervals have the same color.

Proposition 9. *The interval coloring problem is equivalent to a restricted case of the point wrt. wedges coloring problem, where we care only about the wedges with apex on the line L defined by $y = -x$.*

Proof. Consider the natural bijection of the real line and L. Associate to every point p of L the wedge with apex p and associate with every interval $I = (x_1, -x_1), (x_2, -x_2)$ of L the point $(x_1, -x_2)$. It is easy to see that $p \in I$ if and only if the point associated to I is contained in the wedge associated to p. □

Corollary 10. *Any upper bound on the number of colors necessary to (online) color wedges in the plane is also an upper bound for the number of colors necessary to (online) color intervals in* $\mathbb{R}$*.*

Also the lower bounds of Theorem 2 and of Proposition 4 follow for intervals easily by either repeating the proofs for intervals or by Observation 1:

Observation 1. *The proofs of Theorem 2 and of the first part of Proposition 4 can be easily modified such that all the relevant wedges have their apex on the line* $y = -x$*.*

In particular, we have the following.

Corollary 11. *There is a method to online color* N *intervals in* $\mathbb{R}$ *using* $\Theta(\log N/k)$ *colors such that for every point* x*, contained in at least* k *intervals, there exist two intervals containing* x *of different colors.*

As we have seen the results about intervals follow in a straightforward way from the results about wedges. Thus all the statements we proved hold for online coloring wedges, also hold for intervals, however, it seems unlikely that the exact bounds are the same. Thus, we would be happy to see (small) examples where there is a distinction. As the next section shows, there is a difference between the exact bounds for quasi-online coloring wedges and intervals.

3 Quasi-online coloring intervals

A quasi-online coloring of an ordered collection of intervals $\{I_t\}_{t=1}^n$, is a coloring ϕ such that for every k, the sub-collection $\{I_t\}_{t=1}^k$ is properly colored under ϕ.

We exploit an idea used in [11]; instead of online coloring the intervals we online build a labelled acyclic graph (i.e., a forest) with the following properties. Each interval will correspond to a vertex in this graph (there might be other vertices in the graph as well). The final coloring of the intervals will then be generated from this graph. In particular, to define a two-coloring, we will assign each edge in the forest one of two labels, "different" or "same". For an arbitrary coloring of exactly one vertex in each component (tree) of the graph, there is a unique extension to a coloring of the whole graph compatible with the labelling, i.e., such that each edge labelled "same" is adjacent to vertices of the same color and each edge labelled "different" is adjacent to vertices of different colors. In [11] all the edges were labelled "different" so it was actually a simpler variant of our current scheme. As we will see, this idea can also be generalized to more than two colors.

We denote the color of an interval I by $\phi(I)$, the left (resp. right) endvertex of I by $l(I)$ (resp. by $r(I)$). These vertices are real numbers, and so they can be compared.

Theorem 12. *Any finite family of intervals on the line can be colored quasi-online with* 2 *colors such that at any time any point contained by at least* 3 *of these intervals is not monochromatic.*

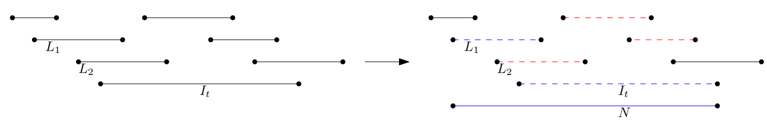

Figure 1: A general step in the proof of Theorem 12

Proof. Let $\{I_t\}_{t=1}^n$ be the given enumeration of the intervals to be quasi-online colored. We first build the forest and then show that the coloring defined by this forest works. As we build the forest we will maintain also a set of intervals, called the active intervals (not necessarily a subset of the given set of intervals). At any time t the vertices of the actual forest correspond to the intervals of $\{I_t\}_{t=1}^n$ and the set of current or past active intervals. The set of active intervals will change during the process, but we maintain that the following properties hold any time.

1. Every point of the line is covered by at most two active intervals.
2. No active interval contains another active interval.
3. A point is either forced by the labelling to be contained in original intervals of different colors or it is contained in the same number of active intervals as original intervals, and additionally the labelling forces these original intervals to have the same colors as these active intervals.
4. Each tree in the forest contains exactly one vertex that corresponds to an active interval.

The last property ensures that a coloring of the active intervals determines a unique coloring of all the intervals which is compatible with the labelling of the forest.

Note that in the third property one or two of the original intervals can actually coincide with one or two of the active intervals.

For the first step we simply make the first interval active; our forest will consist of a single vertex corresponding to this interval. In general, at the beginning of step t, we have a list of active intervals, $\mathcal{J}_{t-1}$. Consider the t^{th} interval, I_t. If I_t is covered by an active interval, $J \in \mathcal{J}_{t-1}$, then we add I_t to the forest and connect it to J with an edge labelled "different". Note that there is at most one such active interval. If there is no active interval containing I_t, we add I_t to the set of active intervals and also add a corresponding vertex to the forest. Now if there are active intervals contained in I_t, these are all deactivated (removed from the set of active intervals) and each is connected to I_t in the graph with an edge labelled "different". Note that this way any point covered by these inactivated intervals will be covered by intervals of both colors.

It remains to ensure that no point is contained within three active intervals. If there still do exist such points, by induction they must be contained within I_t. Let L_1 and L_2 be the (at most) two active intervals covering $l(I_t)$ such that $l(L_1) < l(L_2)$ (if both of them exist.) Similarly, let R_1 and R_2 be the (at most) two active intervals covering $r(I_t)$ such that $l(R_1) < l(R_2)$ (if both of them exist.) We note that the L_i and R_j cannot be the same, as such an interval would cover I_t. Also, no other active intervals can intersect I_t, as they would necessarily be contained in I_t. Without loss of generality we can assume that both L_1 and L_2 exist. If R_1 and R_2 also both exist, deactivate L_1, L_2, I_t, R_1 and R_2 and activate a new interval $N = L_1 \cup I_t \cup R_2$ (and add a corresponding vertex to the graph). In the graph, connect L_1, I_t and R_2 to N

with edges labelled "same". Connect L_2 and R_1 to N with edges labelled "different". Otherwise, if at most one active edge contains $r(I_t)$ we deactive L_1 and L_2 and connect these to the new interval $N = L_1 \cup I_t$ (again with edges labelled "same" and "different", respectively), also we deactive I_t and connect it to N with an edge labelled "same". Figure 3 is an illustration of this case when the active interval N is assigned color blue and deactivated intervals are shown with dashed lines.

This way within a given step, any point which is contained in (at least) two intervals deactivated during the step, is forced by the labelling to be contained in intervals of different colors. For any other point v the number of original intervals containing v remains the same as the number of active intervals covering v (both remains the same or both increases by 1). The first three properties were maintained and also it is easy to check that the graph remains a forest such that in each component there is a unique active interval.

At the end of the process any coloring of the final set of active intervals extends to a coloring of all the intervals (compatible with the labelling of the graph). We have to prove that for this coloring at any time any point contained by at least 3 of these intervals is not monochromatic. By induction at any time $t < n$ the coloring is compatible with the graph at that time, thus by induction any point contained by at least 3 of these intervals is not monochromatic. Now at time n, if the active intervals are colored, the extension (by induction) is such that every point not in I_n is either covered by at most two original intervals or it is covered by intervals of both colors. On the other hand, from the way we defined the graph, we can see that points covered by I_n and contained in at least 3 intervals are covered by intervals of both colors as well. Indeed, by the properties maintained, if a point v is not covered by intervals of both colors than it is covered by as many active intervals as original intervals. Yet, no point is covered by more than 2 active intervals at any time, thus v is covered by no more than 2 active and thus no more than 2 original intervals. □

Theorem 13. *Any finite family of intervals on the line can be colored quasi-online with* 3 *colors such that at any time any point contained by at least* 2 *of these intervals is not monochromatic.*

Proof. We proceed similarly to Theorem 12. In particular, we require the same four properties from active intervals, although the second two need some modifications. Now instead of a labelled graph we define rules of the following form: some interval I (original or auxiliary) gets a different color from at most two other intervals J_1, J_2. We say that I depends from J_1, J_2, otherwise I is independent. If there is an order on the intervals such that an interval depends only on intervals later in this order then starting with any coloring of the independent intervals and then coloring the dependent ones from the last going backwards we can naturally extend this coloring to all the intervals such that the coloring is compatible with the rules (i.e. I gets a color different from the color of J_1, J_2 for all dependent triples). For a representation with directed acyclic graphs - showing more clearly the similarities with the previous proof - see the proof of Theorem 14.

1. Every point of the line is covered by at most two active intervals.
2. No active interval contains another active interval.
3. A point of the line is either forced by the rules to be contained in original intervals of different colors or it is contained in the same number of active intervals as

original intervals, and additionally the rules force these original intervals to have the same colors as these active intervals.

4. An interval is independent if and only if it is an active interval.

The first two properties ensure the following structure on the set of active intervals. Define a chain as a sequence of active intervals such that everyone intersects the one before and after it in the chain. The set of active intervals can be partitioned into disjoint chains. The last property guarantees that any coloring of the active intervals extends naturally and uniquely to a coloring of all the intervals which is compatible with the rules.

We will define the rules such that if we start by a proper coloring of the active intervals then the extension is a quasi-online coloring (as required by the theorem) of the original set of intervals. Note that in the previous proof we started with an arbitrary coloring of the active intervals, which was not necessarily proper, thus now we additionally have to take care that a proper coloring of the active intervals extends to a coloring which is a proper coloring of the active intervals at any previous time as well.

In the first step we add I_1 and activate it. In the induction step we add I_t to the set of active intervals. If I_t is covered by an active interval or by two intervals of a chain, then we deactivate I_t and the rule is that we give a color to it differing from the color(s) of the interval(s). If I_t does not create a triple intersection, it remains activated. Otherwise, denote by L (resp. R) the interval with the leftmost left end (resp. rightmost right end) that covers a triple covered point. We distinguish two cases.

Case i) If I_t is not covered by one chain, then either L or R is I_t, or L and R are not in the same chain. In either case we deactivate all intervals covered by $N = L \cup I_t \cup R$, except for L, I_t and R. The rule to color the now deactivated intervals is that they get a color different from I_t, in an alternating way along their chains starting from L and R.

It is easy to check that the four properties are maintained.

Given a proper coloring of the active intervals at step n by our rules it extends to a proper coloring of the active intervals in the previous step. Thus by induction at any time $t < n$ for any point v it is either covered by differently colored intervals or it is covered by at most one interval. For time n it is either covered by differently colored intervals or it is covered by as many original intervals as active intervals, and they have the same set of colors (by the third property). As the coloring was proper on the active intervals, v is either covered by two original intervals and then two active intervals which have different colors, thus the original intervals have different colors as well, or v is covered by at most one active and thus by at most one original interval.

Case ii) If I_t is covered by one chain, then L and R both differ from I_t. We deactivate all intervals covered by $L \cup I_t \cup R$ (including I_t), except for L and R. Notice that apart from I_t these intervals are all between L and R in this chain.

If we deactivated an odd number of intervals this way (so an even number from the chain), then we insert the new active interval L' that we get from L by prolonging the right end of L such that L' and R intersect in an epsilon short interval. We deactivate L and the rule is to color it the same as we color L'. The rule to color the deactivated I_t is to color it differently from the color of L' (or, equivalently, L) and R. The rule to color the deactivated intervals of the chain is to color them in an alternating way

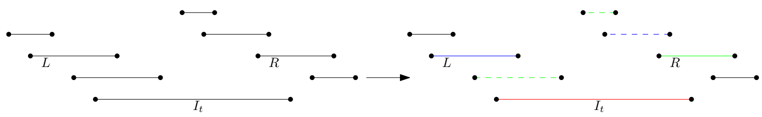

Figure 2: Case i) of Theorem 13

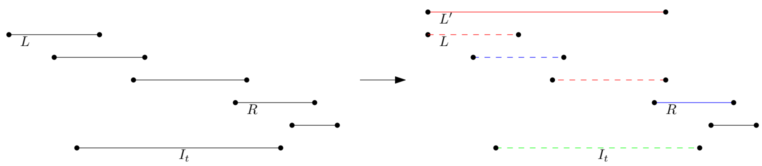

Figure 3: Case ii), odd subcase of Theorem 13

using the colors of L and R (in a final proper coloring of the active intervals they get different colors as L' and R intersect). If we deactivated an even number of intervals this way (so an odd number from the chain), then we deactivate L and R as well and add a new active interval $N = L \cup I_t \cup R$. The rule to color the deactivated I_t is to color it differently from the color of N. The rule to color the deactivated intervals of the chain is to color them in an alternating way using the color of N (L and R get this color) and the color that is different from the color of N and I_t.

It is easy to check that the four properties are maintained. Also, similarly to the previous case, it can be easily checked that if we extend a proper coloring of the active intervals then for its extension it is true at any time (for time $t < n$ by induction, otherwise by the way we defined the rules) that every point is either covered by at most one original interval or it is covered by intervals of different colors. □

Theorem 14. *Colorings guaranteed by Theorem 12 and Theorem 13 can be found in* $O(n \log n)$ *time.*

Proof. Instead of a rigorous proof we provide only a sketch, the easy details are left to the reader. In both algorithms we have n intervals, thus n steps. In each step we define a bounded number of new active intervals, thus altogether we have cn regular and active intervals. We always maintain the (well-defined) left-to-right order of the active intervals. Also we maintain an order of the (active and regular) intervals such that an interval's color depends only on the color of one or two intervals' that are later in this order. This order can be easily maintained as in each step the new interval and the new active intervals come at the end of the order. We also save for each interval the one or two intervals which it depends on. This can be imagined as the intervals represented by vertices on the horizontal line arranged according to this order and an acyclic directed graph on them representing the dependency relations, thus each edge goes backwards and each vertex has indegree at most two (at most one in the first algorithm, i.e. the graph is a directed forest in that case). In each step we have to update the order of active intervals and the acyclic graph of all the intervals, this can be done in $c \log n$ time plus the time needed for the deletion of intervals from the order. Although the latter can be linear in a step, yet altogether during the whole process it remains cn, which is still ok. At the end we just color the vertices one by one from right to left following the rules, which again takes only cn time. Altogether this is $cn \log n$ time. □

These problems are equivalent to (offline) colorings of bottomless rectangles in the plane. Using this notation, Theorem 13 and Theorem 12 were proved already in [14] and [13], yet those proofs are quite involved and they only give quadratic time algorithms, so these results are improvements regarding simplicity of proofs and efficiency of the algorithms.

We note that the algorithms in [14] and [13] proceed with the intervals in backwards order and the intervals are colored immediately (in each step many of them are also recolored), this might be a reason why a lot of recolorings are needed there (which we don't need in the above proofs), adding up to quadratic time algorithms (contrasting the near-linear time algorithms above).

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