On weakly intersecting pairs of sets

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Abstract

Let $\mathcal{F}$ be a family of pairs of sets. We call $\mathcal{F}$ a (k,l)-set system if for $(A,B) \in \mathcal{F}$ we have that $|A| = k$, $|B| = l$, and $A \cap B = \emptyset$. Furthermore, $\mathcal{F}$ is weakly intersecting if for any $(A_i, B_i), (A_j, B_j) \in \mathcal{F}$ with $i \neq j$ we have that $A_i \cap B_j$ and $A_j \cap B_i$ are not both empty. We investigate the maximum possible size of weakly intersecting (k,l)-set systems. We improve on the lower bound of Tuza [10] and compute the exact value for $k = l = 2$.

1 Introduction

Let $\mathcal{F}$ be a family of pairs of sets. We call $\mathcal{F}$ a (k,l)-set system if for $(A,B) \in \mathcal{F}$ we have that $|A| = k$, $|B| = l$, $A \cap B = \emptyset$. The following classical result on intersecting pairs of sets is due to Bollobás [5]. Let $\mathcal{F}$ be a (k,l)-set system with the intersecting property, i.e., for $(A_i, B_i), (A_j, B_j) \in \mathcal{F}$ with $i \neq j$ we have that $A_i \cap B_j$ and $A_j \cap B_i$ are both non-empty. The maximum possible size of such a set system is $\binom{k+l}{k}$ and this is sharp. Interestingly, the same upper bound holds even if for $(A_i, B_i), (A_j, B_j) \in \mathcal{F}$ we only require that $A_i \cap B_j \neq \emptyset$ for $i < j$, as was shown in [6]. Several further generalizations were investigated in [1, 7, 8, 9]. In this paper we consider the following variant of the problem, introduced by Tuza [10] (for more details on the history we refer to the surveys [11, 12].)

Definition 1.1. *Let $\mathcal{F}$ be a (k,l)-set system. $\mathcal{F}$ is weakly intersecting if for $(A_i, B_i), (A_j, B_j) \in \mathcal{F}$ with $i \neq j$ we have that $A_i \cap B_j$ and $A_j \cap B_i$ are not both empty.*

The goal of this paper is to investigate the maximum possible size of such a system, which we denote by $g(k,l)$. In Section 2 we give a construction based on lattice paths and prove that $\liminf_{k+l\to\infty} g(k,l)/\binom{k+l}{k} \geq 2 - o(1)$.[1] This improves on the lower bound in [10]. In Section 3 we recall the known upper bounds and in Section 4 we present some computational results for small values of k and l, in particular, we show that $g(2,2) = 10$. We conclude by posing several interesting open problems.

[1]Whenever we use $o(1)$, we mean that the number tends to 0 as $k + l$ tends to infinity.

2 Lower bounds

We start from the following two claims made by Tuza [10].

Claim 2.1. $g(k,1) \geq 2k+1$.

Proof. Let $B_i = \{i\}$ and $A_i = \{i + j \mod (2k+1) \mid 1 \leq j \leq k\}$ for $i = 0, \ldots, 2k$. $\square$

Claim 2.2. $g(k,l) \geq g(k-1,l) + g(k,l-1)$.

Proof. Suppose we have a construction $\{(A_i, B_i)\}$ of cardinality $g(k-1,l)$ and another construction $\{(A'_j, B'_j)\}$ of cardinality $g(k,l-1)$. Let x be an element not contained in any of these sets. Then, $\{(A_i \cup \{x\}, B_i)\} \cup \{(A'_j, B'_j \cup \{x\})\}$ gives a construction for a (k,l)-set system of cardinality $g(k-1,l) + g(k,l-1)$. $\square$

From these claims one can easily obtain the following corollary.

Corollary 2.3. $g(k,l) \geq 2\binom{k+l}{k} - \binom{k+l-2}{k-1} = \left(2 - \frac{kl}{(k+l)(k+l-1)}\right)\binom{k+l}{k}$.

Proof. We prove by induction. Let $\widetilde{g}(k,l) = 2\binom{k+l}{k} - \binom{k+l-2}{k-1}$. From Claim 2.1 we have that $g(k,l) \geq \widetilde{g}(k,l)$ for all $(k,1)$ and $(1,l)$ which settles the base case. The inductive step follows from Claim 2.2 and the fact that $\widetilde{g}(k,l) = \widetilde{g}(k-1,l) + \widetilde{g}(k,l-1)$. $\square$

We would like to point out that from this we get that $\lim_{k\to\infty} \widetilde{g}(k,k)/\binom{2k}{k} = \frac{7}{4}$ and not 2 as was claimed in [10]. In what follows, we give an explicit construction that proves that the above ratio for the lower bound is at least $2 - o(1)$ for all k and l as $k+l$ tends to infinity.

2.1 A construction using lattice paths

Now we describe the construction of a (k,l)-set system $\mathcal{F}$ for arbitrary k and l using lattice paths. Let $m = 2k + 2l - 1$. We will construct $\mathcal{F} = \{(A,B) : |A| = k,\ |B| = l,\ A, B \subset \{0, \ldots, m-1\}\}$. Denote by $\mathcal{L}(k,l)$ the set of lattice paths on a $k \times l$ grid from $(0,0)$ to (k,l), where each move is to the right or up, and the path is strictly below the diagonal except at the two endpoints. We will identify a path $\pi \in \mathcal{L}(k,l)$ with its binary representation $\pi \in \{0,1\}^{k+l}$ using 0 and 1 for steps to the right and up, resp. For each path $\pi \in \mathcal{L}(k,l)$ we create the following m set pairs $(A_{\pi,i}, B_{\pi,i})$ where $i = 0, \ldots, m-1$. Let $A_{\pi,i} = \{i + t \mod m \mid 1 \leq t \leq k+l,\ \pi(t) = 0\}$ and $B_{\pi,i} = \{i + t \mod m, \mid 1 \leq t \leq k+l,\ \pi(t) = 1\}$.

We have to show that the obtained $\mathcal{F}$ indeed satisfies Definition 1.1. Clearly, it is a (k,l)-set system by construction. To show the weakly intersecting property, first observe that $A_{\pi,i} \cap B_{\sigma,i} \neq \emptyset$ for any $\pi \neq \sigma$ as the paths are different and hence $\exists t : \pi(t) = 0,\ \sigma(t) = 1$. If $i \neq j$ then $A_{\pi,i} \cap B_{\sigma,j} \neq \emptyset$ or $A_{\sigma,j} \cap B_{\pi,i} \neq \emptyset$ follows from the fact that any proper suffix of a lattice path in $\mathcal{L}(k,l)$ cannot equal the prefix of another (or even the same) lattice path, and hence $\exists t : \pi(t) \neq \sigma(t+i-j)$. One way to see this is to consider the ratio of the right and up steps: in the prefix this ratio has to be greater than k/l whereas in the suffix it has to be strictly less than that.

Remark 2.4. *We note that for* $\gcd(k,l) = 1$ *this* (k,l)*-set system is* maximal *in the sense that we cannot add another pair of sets to it without violating the weakly intersecting property.*

Unfortunately, there is no simple closed formula for $|\mathcal{L}(k,l)|$ for general (k,l). However, for relative primes k and l we have the following theorem.

Theorem 2.5 (Bizley [3])**.** $|\mathcal{L}(k,l)| = \binom{k+l}{k}/(k+l)$ *if* $\gcd(k,l) = 1$.

From Theorem 2.5 and the construction described above we obtain a better lower bound than in Corollary 2.3.

Theorem 2.6. $g(k,l) \geq (2k+2l-1)|\mathcal{L}(k,l)| = (2k+2l-1)\binom{k+l}{k}/(k+l)$ *if* $\gcd(k,l) = 1$.

Corollary 2.7. $g(k,k-1) \geq (2 - \frac{1}{2k-1}) \cdot \binom{2k-1}{k}$.

Corollary 2.8. $g(k,k) \geq (2 - \frac{1}{2k-1}) \cdot \binom{2k}{k}$.

Proof. Claim 2.2 and Corollary 2.7 gives $g(k,k) \geq 2g(k,k-1) = (2 - \frac{1}{2k-1}) \cdot \binom{2k}{k}$. $\square$

Applying this trick multiple times we obtain our main theorem, a stronger lower bound for all (k,l).

Theorem 2.9. $g(k,l) \geq (2 - o(1))\binom{k+l}{k}$.

Proof. The idea is to use Claim 2.2 repeatedly until we decrease $k+l$ to a prime p for which we automatically have $\gcd(p-q,q) = 1$ for $0 < q < p$ and then invoke Theorem 2.6.

Baker et al. proved in [4] that for $x > x_0$ the interval $[x - x^{0.525}, x]$ contains a prime. We will use this result for $\mathbf{x} := \mathbf{k} + \mathbf{l}$. If either k or l is smaller than $x^{0.525}$ then by Corollary 2.3 we have $g(k,l)/\binom{k+l}{k} \geq 2 - \frac{x \cdot x^{0.525}}{x(x-1)} \geq 2 - o(1)$. Otherwise, we will use the result that there is a prime p such that $0 \leq x - p \leq x^{0.525} \leq k,l$. By applying Claim 2.2 several times and then Theorem 2.6 we obtain

$$g(k,l) \geq \sum_{i=0}^{x-p} \binom{x-p}{i} g(k-(x-p)+i,\ l-i) = \sum_{i=0}^{k+l-p} \binom{k+l-p}{i} g(p-(l-i),\ l-i) \geq$$

$$\geq \sum_{i=0}^{k+l-p} \binom{k+l-p}{i} (2p-1) \binom{p}{l-i} / p = \frac{2p-1}{p} \binom{k+l}{l}$$

and since $(2p-1)/p \geq 2 - 1/(x - x^{0.525}) = 2 - o(1)$, we have proven the theorem. $\square$

3 Upper bounds

For the sake of completeness we also mention the upper bounds which appeared in [10].

Claim 3.1. $g(k,1) = 2k+1$.

For general k and l we do not have matching upper bounds. In fact, the following theorem (also appeared as an exercise in [2]) is the best known.

Theorem 3.2 (Tuza, [10])**.** $g(k,l) < \frac{(k+l)^{k+l}}{k^k l^l}$.

4 Proving bounds using computer programs

Next we present a straightforward method for computing $g(k,l)$ for fixed k and l with the help of a computer program. If we know the size (or an upper bound on the size) of the ground set we can find a maximum size weakly intersecting (k,l)-set system as follows. First, generate all possible (k,l)-set pairs. Then create a graph, whose vertices represent set pairs and edges are drawn between vertices if the corresponding set pairs weakly intersect. Finally, find a maximum clique in this graph (this corresponds to a maximal set system).

Let us denote the size of the smallest ground set on which it is possible to realize a weakly intersecting (k,l)-set system with N set-pairs by $n(k,l,N)$. We are only interested in the case where $N > \binom{k+l}{l}$, consequently $n(k,l,N) > k+l$.

Claim 4.1. $\binom{n(k,l,N)}{2} \leq kl \cdot N$.

Proof. Suppose we have a construction of size N on a set of size n and $\binom{n}{2} > kl \cdot N$. Define a graph on the ground set in the following way. Connect x and y if there is an i such that A_i contains one of them and B_i the other one. By our assumption, there must be a pair of points that are not connected to each other. Contracting these points gives a smaller ground set, where still any two set-pairs intersect. It might happen that x and y are both in some A_j (or B_j), in this case just add to the shrunken set an arbitrary element that is not in $A_j \cup B_j$, as $n > k+l$ we can choose such an element. $\square$

Theorem 4.2. $g(2,2) = 10$.

Proof. By Corollary 2.8 we have that $g(2,2) \geq 10$. So, we have to prove $g(2,2) \leq 10$. We prove by contradiction. Suppose that there is a weakly intersecting $(2,2)$-set system with $N = 11$ set-pairs. Then Claim 4.1 implies that this has a realization on a ground set with at most 9 elements. We performed an exhaustive search on the ground set of size 9 with our computer program, verifying that the maximum size of a $(2,2)$-set system is 10. $\square$

Unfortunately, the running time of this brute force search grows superexponentially and we could not compute further values. We found a $(3,2)$-set system of size 19, hence $g(3,2) \geq 19$ (which also implies $g(3,3) \geq 38$), thus our lower bound is not always optimal.

We would like to conclude with several interesting questions.

Open problems.

Is $g(k,k) = 2g(k-1,k)$ for all $k \geq 2$?
Is $g(k,l) < 2\binom{k+l}{k}$?
Is $g(k,k) = o(2^{2k})$?

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