Unit disks hypergraphs are three-colorable

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Abstract. We prove that any finite point set $\mathcal{P}$ in the plane can be three-colored such that there is no unit disk containing at least 1025 points, all of the same color.

Keywords: Discrete geometry, Geometric hypergraph coloring, Decomposition of multiple coverings

1 Introduction

Coloring problems for hypergraphs defined by geometric range spaces have been studied extensively in different settings [1–21]. A pair $(\mathcal{P}, \mathcal{S})$, where $\mathcal{P}$ is a set of points in the plane and $\mathcal{S}$ is a family of subsets of the plane (the *range space*), defines a (primal) hypergraph $\mathcal{H}(\mathcal{P}, \mathcal{S})$ whose vertex set is $\mathcal{P}$, and edge set is $\{S \cap \mathcal{P} \mid S \in \mathcal{S}\}$. Given any hypergraph $\mathcal{G}$, a planar realization of $\mathcal{G}$ is defined as a pair $(\mathcal{P}, \mathcal{S})$ for which $\mathcal{H}(\mathcal{P}, \mathcal{S})$ is isomorphic to $\mathcal{G}$. If $\mathcal{G}$ can be realized with some pair $(\mathcal{P}, \mathcal{S})$, where $\mathcal{S}$ is from some family $\mathcal{F}$, then we say that $\mathcal{G}$ is realizable with $\mathcal{F}$.

It is an easy consequence of the properties of Delaunay-triangulations and the Four Color Theorem that the vertices of any hypergraph realizable with disks can be four-colored such that every edge that contains at least two vertices contains two differently colored vertices. But are less colors sufficient if all edges are required to contain at least m vertices for some large enough constant m? The authors settled this question recently [6], showing that three colors are not enough for any m, i.e., for any m, there exists an m-uniform hypergraph that is not three-colorable and that permits a planar realization with disks.

For unit disks in arbitrary position, Pach and Pálvölgyi [16] showed that for any m, there exists an m-uniform hypergraph that is not two-colorable and that permits a planar realization with unit disks. Our main result is showing that for large enough m three colors are sufficient for unit disks.

Theorem 1. *Any finite point set $\mathcal{P}$ can be three-colored such that any unit disk that contains at least 1025 points from $\mathcal{P}$ contains two points colored differently.*

2 Hypergraph Colorings

It is important to distinguish between two types of hypergraph colorings that we will use, the *proper* coloring and the *polychromatic* coloring.

Definition 1. *A hypergraph is* properly k-colorable *if its vertices can be colored with k colors such that each edge contains points from at least two color classes. Such a coloring is called a* proper coloring.

Definition 2. *A hypergraph is* polychromatic k-colorable *if its vertices can be colored with k colors such that each edge contains points from each color class. Such a coloring is called a* polychromatic coloring.

Polychromatic colorability was studied for many geometric families. For hypergraphs determined by pseudohalfplanes (defined as the regions on one side of each pseudoline in some pseudoline arrangement) the following is known.

Theorem 2 (Keszegh-Pálvölgyi [12]). *Given a finite collection of points and pseudohalfplanes, the points can be k-colored such that every pseudohalfplane that contains at least $2k-1$ points contains all k colors.*

Polychromatic colorability is a much stronger property than proper colorability. Any polychromatic k-colorable hypergraph is proper 2-colorable. We generalize this trivial observation to the following statement about unions of polychromatic k-colorable hypergraphs.

Theorem 3. *Let $\mathcal{H}_1 = (V, E_1), \ldots, \mathcal{H}_{k-1} = (V, E_{k-1})$ be hypergraphs on a common vertex set V. If $\mathcal{H}_1, \ldots, \mathcal{H}_{k-1}$ are polychromatic k-colorable, then the hypergraph $\bigcup_{i=1}^{k-1} \mathcal{H}_i = (V, \bigcup_{i=1}^{k-1} E_i)$ is proper k-colorable.*

Proof. Let $c_i : V \to \{1, \ldots, k\}$ be a polychromatic k-coloring of $\mathcal{H}_i$. Choose $c(v) \in \{1, \ldots, k\}$ such that it differs from each $c_i(v)$. We claim that c is a proper k-coloring of $\bigcup_{i=1}^{k-1} \mathcal{H}_i$. To prove this, it is enough to show that for every edge $H \in \mathcal{H}_i$ and for every color $j \in \{1, \ldots, k-1\}$, there is a $v \in H$ such that $c(v) \neq j$. We can pick $v \in H$ for which $c_i(v) = j$. This finishes the proof.

Theorem 3 is sharp in the sense that for every k there are $k-1$ polychromatic k-colorable hypergraphs such that their union is not properly $(k-1)$-colorable.

3 Proof of Theorem 1

Let $\mathcal{P}$ denote the points and let $\mathcal{D}$ denote the unit (radius) disks that contain at least 1025 points from $\mathcal{P}$.

The first step of the proof is a classic divide and conquer idea [15]. Divide the plane into a grid of squares of side length $\frac{1}{\sqrt{10}} \approx 0.31$ such that no point of $\mathcal{P}$ falls on the boundary of a grid square. Since a square of side length two intersects at most eight rows and eight columns of the grid, each unit disk intersects at most 64[1] squares. Let $D \in \mathcal{D}$ be one of the unit disks. Since D contains at least

[1] In fact less, but we are not trying to optimize our constants.

$1025 = 64 \cdot 16 + 1$ points from $\mathcal{P}$, by the pigeonhole principle there is a square S such that $S \cap D$ contains at least 17 points from $\mathcal{P}$.

Hence it is enough to show the following theorem. Applying it separately for the points in each square of the grid provides a proper three-coloring of the whole point set.

Lemma 1. *Suppose $\mathcal{P}$ is a finite point set inside a square of side length $\frac{1}{\sqrt{10}}$. Then we can color the points of $\mathcal{P}$ by three colors such that any unit disk, that contains at least 17 points from $\mathcal{P}$, will contain points from all three colors.*

Proof. Since $2 \cdot (\frac{1}{\sqrt{10}})^2 < 1$, if the center of a unit disk lies in the square, then the disk contains the whole square. As we will use more than one color to color the points in the square, such disks cannot be monochromatic. The sidelines of the square divide the plane into nine regions. Denote the unbounded closed quadrant regions by Q_1, Q_2, Q_3, Q_4 and the unbounded open half-strip regions by S_1, S_2, S_3, S_4, numbered in a clockwise order, according to Figure 1. We need to assure that no matter in which of these eight regions the center of a unit disk lies, it is not monochromatic.

Let ℓ_x be a horizontal halving line for $\mathcal{P}$, that is, a horizontal line such that both (closed) halfplanes bounded by ℓ_x contain at least $\frac{|\mathcal{P}|}{2}$ points. Similarly, let ℓ_y be a vertical halving line for $\mathcal{P}$ and let O denote the intersection of ℓ_x and ℓ_y. These lines divide the square into four (closed) rectangular regions R_1, R_2, R_3, R_4, indexed according to Figure 1. The usefulness of this further subdivision comes from the following observation.

Observation 1 *If the center of a unit disk lies in Q_i and the disk contains O, then the disk contains the whole region R_i.*

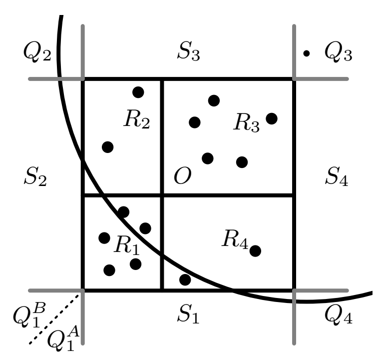

Fig. 1. Regions around a grid square.

We will color the four regions $R_1, \ldots, R_4$ separately, but Observation 1 reduces the number of disks that have to be considered for each region.

Let $\mathcal{D}_i \subset \mathcal{D}$ denote the disks that contain at least 5 point from $R_i \cap \mathcal{P}$. We will color the points of R_i with three colors such that for each $D \in \mathcal{D}_i$ the following holds: either $D \cap R_i \cap \mathcal{P}$ is not monochromatic or D contains the whole region R_{i+2} (indexed modulo 4).

By symmetry it is enough to consider R_1. If $|R_1 \cap \mathcal{P}| \leq 4$, then $\mathcal{D}_i$ is empty and we are done. Otherwise we divide the disks in $\mathcal{D}_i$ into three groups. The line of the diagonal from the bottom-left corner of the square to its top-right corner splits Q_1 into two parts, as marked with a dashed line on Figure 1. Denote by Q_1^A the bottom-right part of Q_1 and by Q_1^B its upper-left part. Let $\mathcal{A} \subset \mathcal{D}_1$ be the disks whose center lies in $Q_1^A \cup S_1 \cup Q_4 \cup S_4$. Let $\mathcal{B} \subset \mathcal{D}_1$ be the disks whose center lies in $Q_1^B \cup S_2 \cup Q_2 \cup S_3$. Let $\mathcal{C} \subset \mathcal{D}_1$ be the disks whose center lies in the closed quadrant Q_3.

If a disk is in $\mathcal{C}$, then it contains O, thus by Observation 1 it also contains the whole region R_3, and the coloring of the points $\mathcal{P} \cap R_3$ will ensure that it cannot be monochromatic. Hence, it is enough to properly three-color the hypergraph $\mathcal{H}(R_1 \cap \mathcal{P}, \mathcal{A} \cup \mathcal{B})$. First we show that both $\mathcal{H}(R_1 \cap \mathcal{P}, \mathcal{A})$ and $\mathcal{H}(R_1 \cap \mathcal{P}, \mathcal{B})$ are realizable with pseudohalfplanes. We use the following geometric lemma.

Lemma 2. *If we take two disks from $\mathcal{A}$, or two disks from $\mathcal{B}$, their boundaries intersect at most once inside R_1.*

Proof. Let $R = \cup_{i=1}^4 R_i$ denote the square and define two trapezoidal regions around R as follows. Denote by X^* the reflection of any region X to the bottom-right corner of the square R. One trapezoid is $(Q_1^A \cup S_1) \cap S_4^*$, and the other is $(Q_1^A \cup S_1)^* \cap S_4$, see the shaded regions on Figure 2. The trapezoids have $45°, 90°$ and $135°$ degree angles and the ratio of their sides is $1 : 1 : 2 : \sqrt{2}$.

Let D_1 and D_2 be two disks from $\mathcal{A}$ and let o_1, o_2 denote their centers. If the boundaries of D_1 and D_2 intersect twice inside R, then the midpont of $o_1 o_2$ falls into R. It is easy to see that this implies that o_1 and o_2 must be located in the shaded regions shown in Figure 2. If we place o_1 outside of the shaded region, then the possible locations for o_2 fall outside of $Q_1^A \cup S_1 \cup Q_4 \cup S_4$. On the other hand if o_1, o_2 are in the shaded region, then D_1 and D_2 contains R as $(\frac{3}{\sqrt{10}})^2 + (\frac{1}{\sqrt{10}})^2 = 1$. This contradicts that their boundaries intersect inside R.

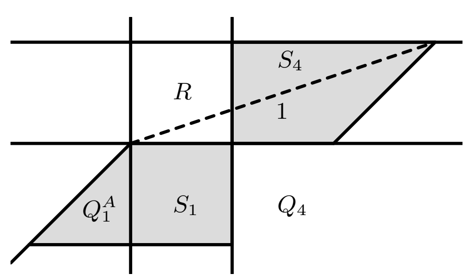

Fig. 2. Locations for two points in $Q_1^A \cup S_1 \cup Q_4 \cup S_4$ whose midpoint lies in R.

A similar argument holds for $\mathcal{B}$, finishing the proof of Lemma 2.

We remark that Lemma 2 holds also for squares of side length $\frac{1}{\sqrt{5}}$ with a more careful argument.

Therefore, $\mathcal{H}(R_1 \cap \mathcal{P}, \mathcal{A})$ and $\mathcal{H}(R_1 \cap \mathcal{P}, \mathcal{B})$ are hypergraphs that can be realized by pseudohalfplanes. By definition each edge in these hypergraphs contains at least 5 vertices. Thus by Theorem 2 they are polychromatic three-colorable, and by Theorem 3, $\mathcal{H}(R_1 \cap \mathcal{P}, \mathcal{A} \cup \mathcal{B})$ is proper three-colorable.

We apply the previous argument for each R_i. To see that the resulting coloring is good, take any disk $D \in \mathcal{D}$. Since D contains at least $17 = 4 \cdot 4 + 1$ points from $\mathcal{P}$, there is a region R_i such that D contains at least 5 points from $R_i \cap \mathcal{P}$, that is $D \in \mathcal{D}_i$. Therefore either D contains two points of different colors in R_i, or D contains the whole region R_{i+2}. Since ℓ_x and ℓ_y are halving lines $|\mathcal{P} \cap R_i| = |\mathcal{P} \cap R_{i+2}|$ (indexed modulo 4). Hence region R_{i+2} contains at least 5 point from $\mathcal{P}$. The points inside R_{i+2} are not monochromatic, hence D is not monochromatic in either case.

4 Concluding Remarks

Let the *m-fat* edges of a hypergraph be those edges whose cardinality is at least m. We can restate Theorem 1 the following way. If $\mathcal{P}$ is a set of point in the plane and $\mathcal{S}$ is a set of unit disks, then the m-fat edges of the hypergraph $\mathcal{H}(\mathcal{P}, \mathcal{S})$ form a hypergraph that is properly three-colorable.

One can consider other geometric families for $\mathcal{S}$. For example, let C be a convex compact set whose boundary is smooth and let $\mathcal{S}$ be a family of translates of C. A small refinement of the argument above shows that there is an $m = m(C)$ such that the m-fat edges of $\mathcal{H}(\mathcal{P}, \mathcal{S})$ form a three-colorable hypergraph. It was shown in [16] that for every smooth compact set C and for every m there is a non-two-colorable hypergraph that can be realized by C. We can show that this result extends to several sets whose boundary is only partly smooth, such as a halfdisk, answering an open problem from [16]. The construction is essentially the same as in [13, 16, 19], using the arrangement shown in Figure 3 for the recursive step.

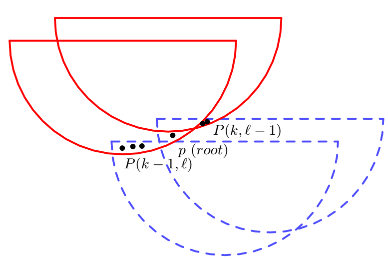

Fig. 3. Recursive step for the non-two-colorable half disk construction.

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