2D-TUCKER is PPAD-complete

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This means that TUCKER is **PPAD – complete**.

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The problem for which the input is n and the description of the Turing-machine and the output is another path ending is called END-OF-THE-LINE.

What is complete?

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From Wiktionary:

complete (comparative more complete, superlative most complete)

1. With everything included.

It was a complete shock when he turned up on my doorstep.

2. Finished; ended; concluded; completed; as, the edifice is complete.

3. (analysis, of a metric space) in which every Cauchy sequence converges

4. (algebra, of a lattice) in which every set with a lower bound has a greatest lower bound

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PPAD – complete means the problem is in **PPAD** and every problem in **PPAD** is reducible to the problem.

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List maintained by Shiva Kintali.

Tucker's lemma

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Theorem (Tucker's lemma)

For any triangulation of the n-dimensional ball that is antipodally symmetric on the boundary and any coloring of the vertices with $+1, -1, +2, \ldots, +n, -n$ *that is also antipodal on the boundary, we have two vertices that are colored by opposite numbers.*

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Definition (2D-TUCKER)

We are given the description of a deterministic Turing machine M *which for every input* $(u, v) \in \{0,1\}^{2n}$ *returns either* 1, -1, 2 *or* -2 *in time* $poly(n)$. *Furthermore, for all* i $M(0, i) = -M(2^n, 2^n - i)$ *and* $M(i, 0) = -M(2^n - i, 2^n)$. *The output (whose existence is guaranteed by Tucker's lemma) is* $(u, v) \in \{0,1\}^{2n}$ *and* $(u', v') \in \{0,1\}^{2n}$ *for which* $|u - u'| \leq 1$, $|v - v'| \leq 1$ *and* $M(u, v) = -M(u', v')$.

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We are given the description of a deterministic Turing machine M which for every input $(u, v) \in \{0,1\}^{2n}$ returns either 1, -1, 2 or -2 in time poly(n). Furthermore, for all i $M(0, i) = -M(2^n, 2^n - i)$ and $M(i, 0) = -M(2^n - i, 2^n)$. The output (whose existence is guaranteed by Tucker's lemma) is $(u, v) \in \{0,1\}^{2n}$ and $(u', v') \in \{0,1\}^{2n}$ for which $|u - u'| \leq 1$, $|v - v'| \leq 1$ and $M(u, v) = -M(u', v')$.

Note that here instead of triangles we use squares.

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It easily follows from the known proofs for Tucker's lemma.

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Chen and Deng proved that 2D-SPERNER is **PPAD – complete**.

We will use a method similar to theirs.

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The color of every vertex in the square will be determined by a finite number of queries to the Turing-machine that determines the graph.

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The color of every vertex in the square will be determined by a finite number of queries to the Turing-machine that determines the graph.

Finding two neighbors with opposite colors will give a path end.

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-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	2
-1	2	2	2	2	2	2	2	2	2	2	2
-1	2	1	1	1	1	1	1	1	1	1	1
-1	2	1									1
-1	2	1									1
-1	2	1		2	2	2	2				1
-1	2	1		2	-1	-1	-1				1
-1	2	1	1	2	-1	-2	-2				1
-1	2	2	2	2	-1	-2					1
-1	-1	-1	-1	-1	-1	-2					1
-2	-2	-2	-2	-2	-2	-2					1
-2	1	1	1	1	1	1	1	1	1	1	1

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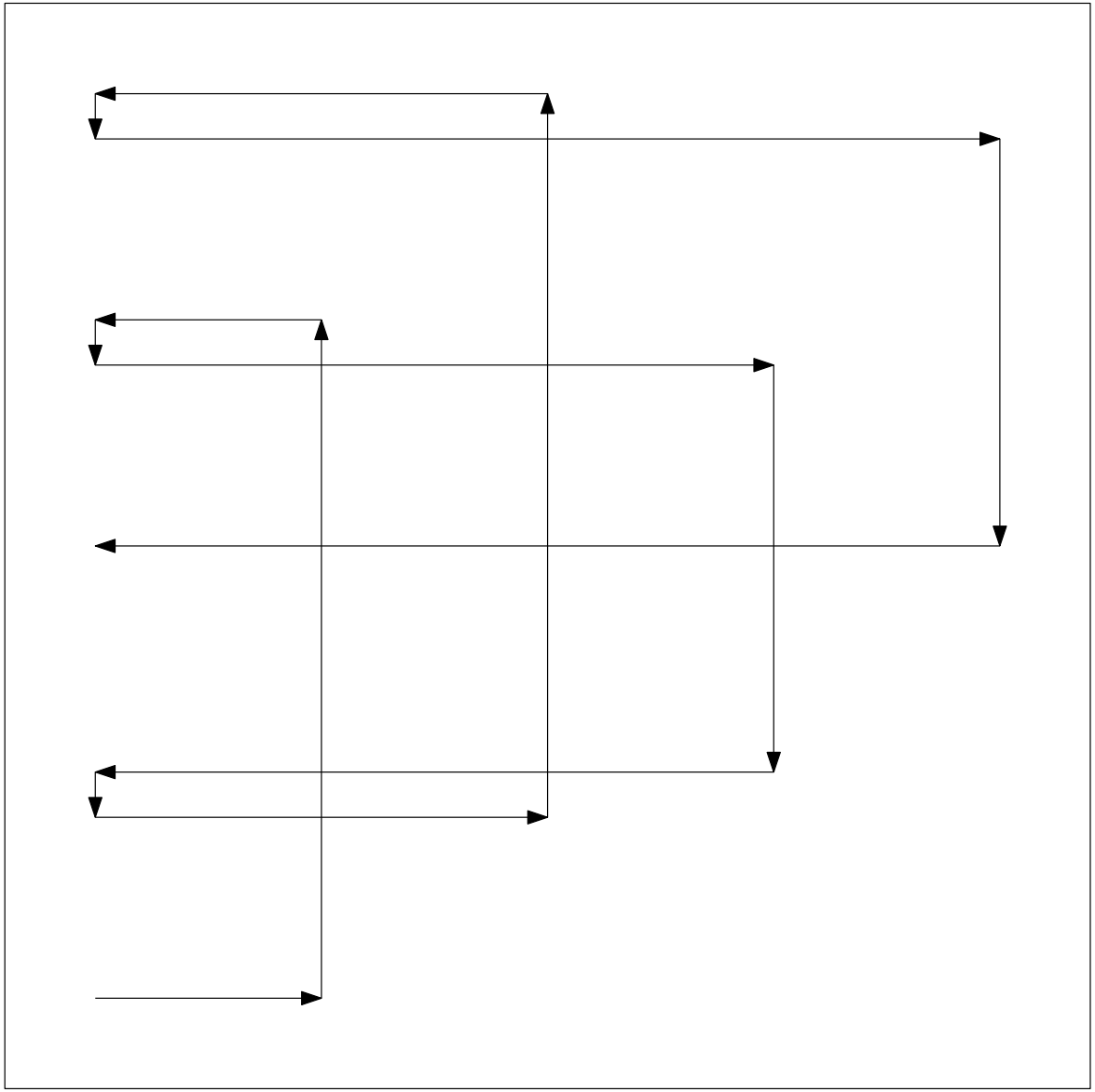

2D-TUCKER is PPAD-complete

	-2	-2	-2	-2	-2	-2	-2	-2
V_j	-1	-1	-1	-1	-1	-1	-1	-2
	2	2	2	2	2	2	-1	-2
						2	-1	-2
						2	-1	-2
						2	-1	-2
						2	-1	-2
	2	2	2	2	2	2	-1	-2
V_i	-1	-1	-1	-1	-1	-1	-1	-2
	-2	-2	-2	-2	-2	-2	-2	-2

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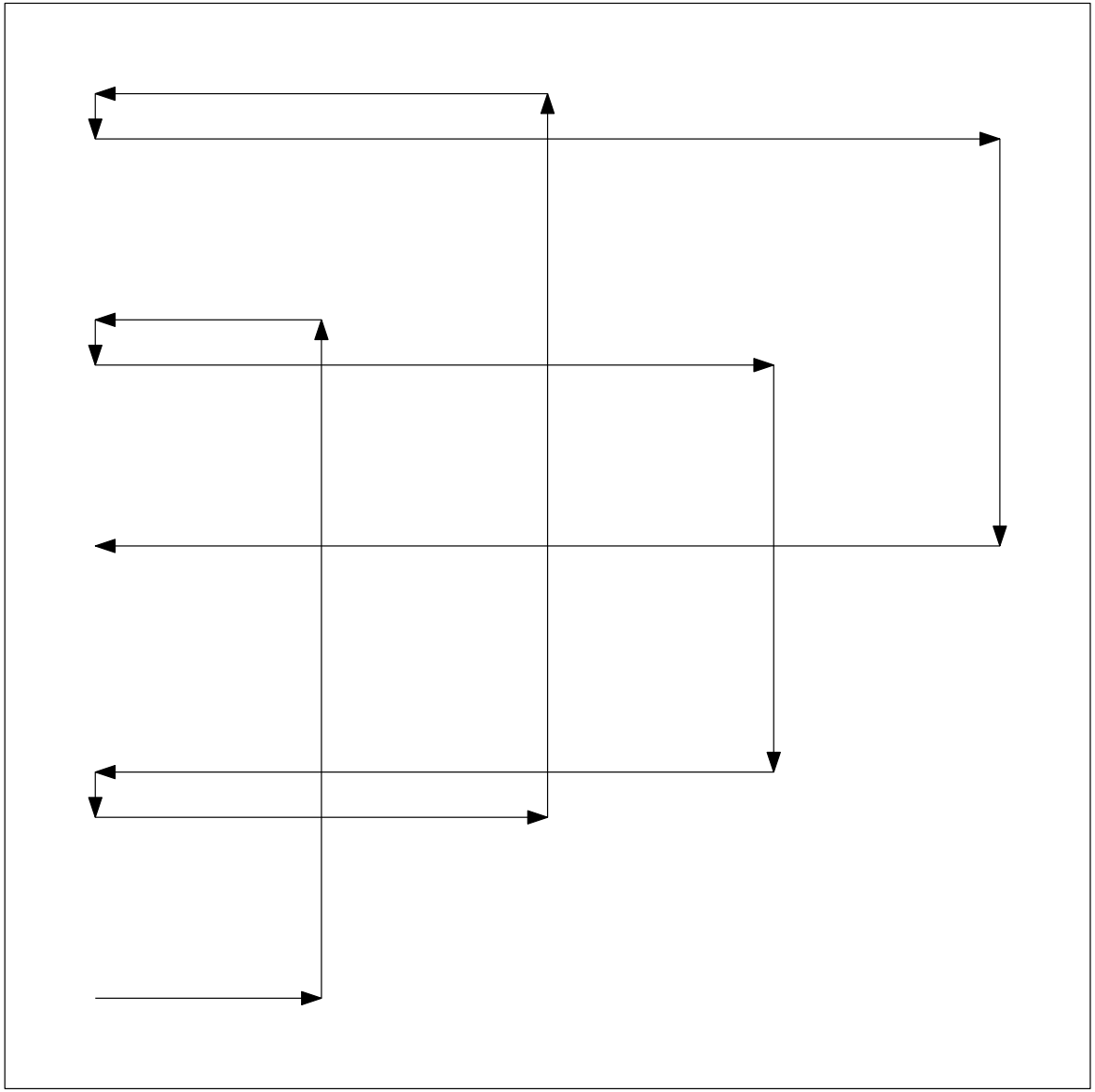

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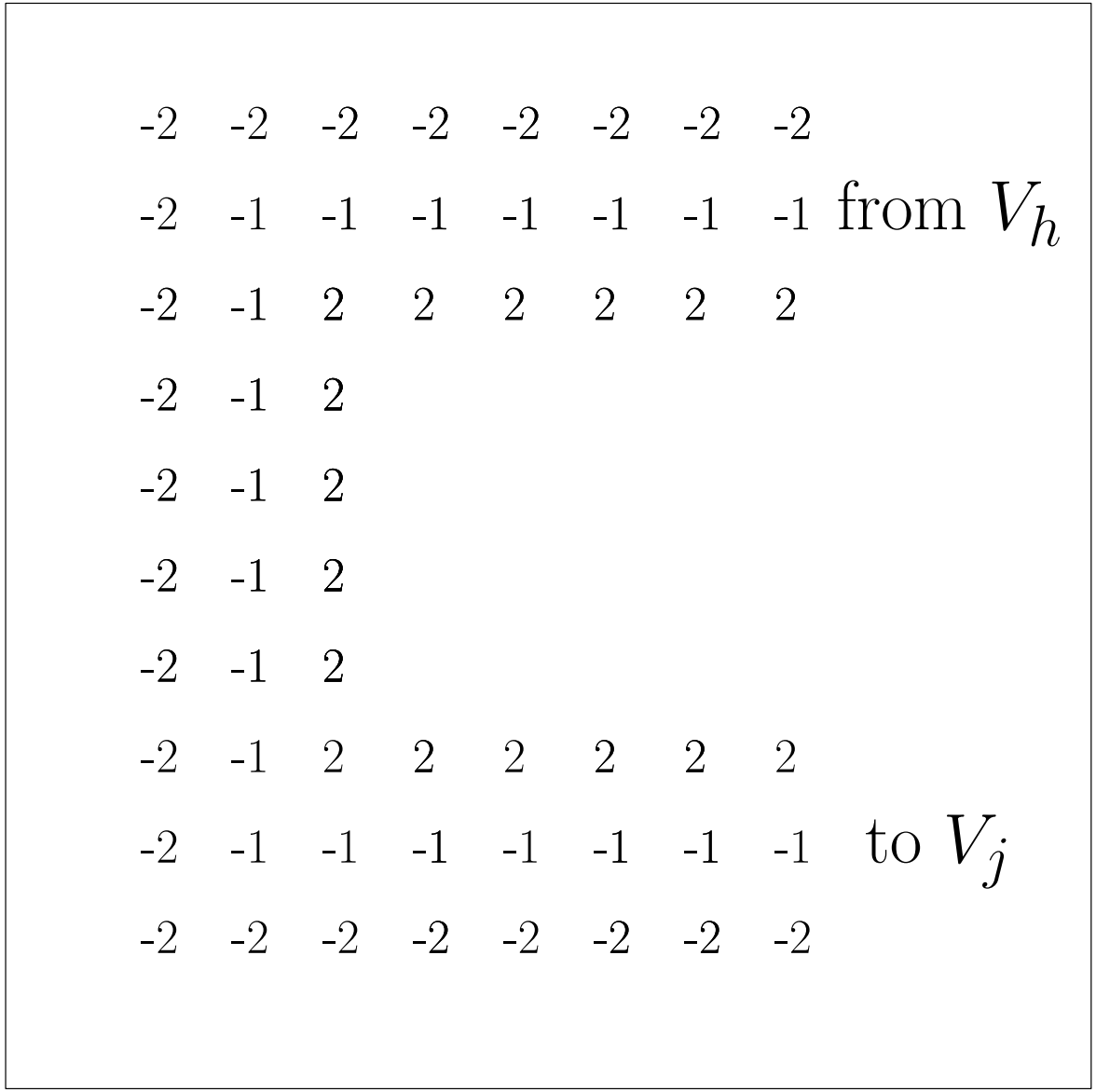

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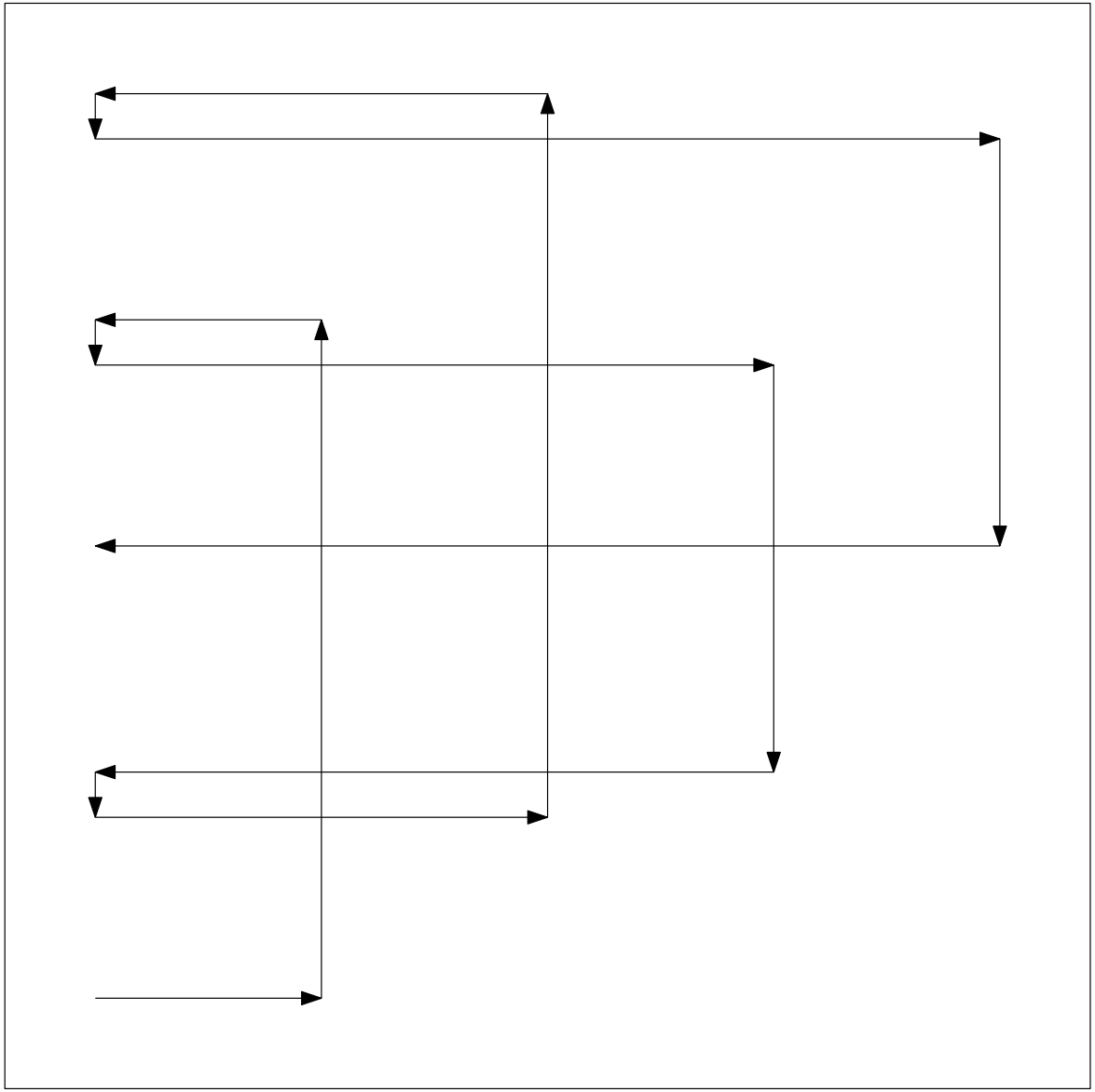

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				-2	-1	2	2	2			
				-2	-1	-1	-1	2			
				-2	-2	-2	-1	2			
						-2	-1	2			
2	2	2	2	2		-2	-1	2	2	2	2
-1	-1	-1	-1	2		-2	-1	-1	-1	-1	-1
-2	-2	-2	-1	2		-2	-2	-2	-2	-2	-2
		-2	-1	2							
		-2	-1	2	2	2					
		-2	-1	-1	-1	2					
		-2	-2	-2	-1	2					
				-2	-1	2					

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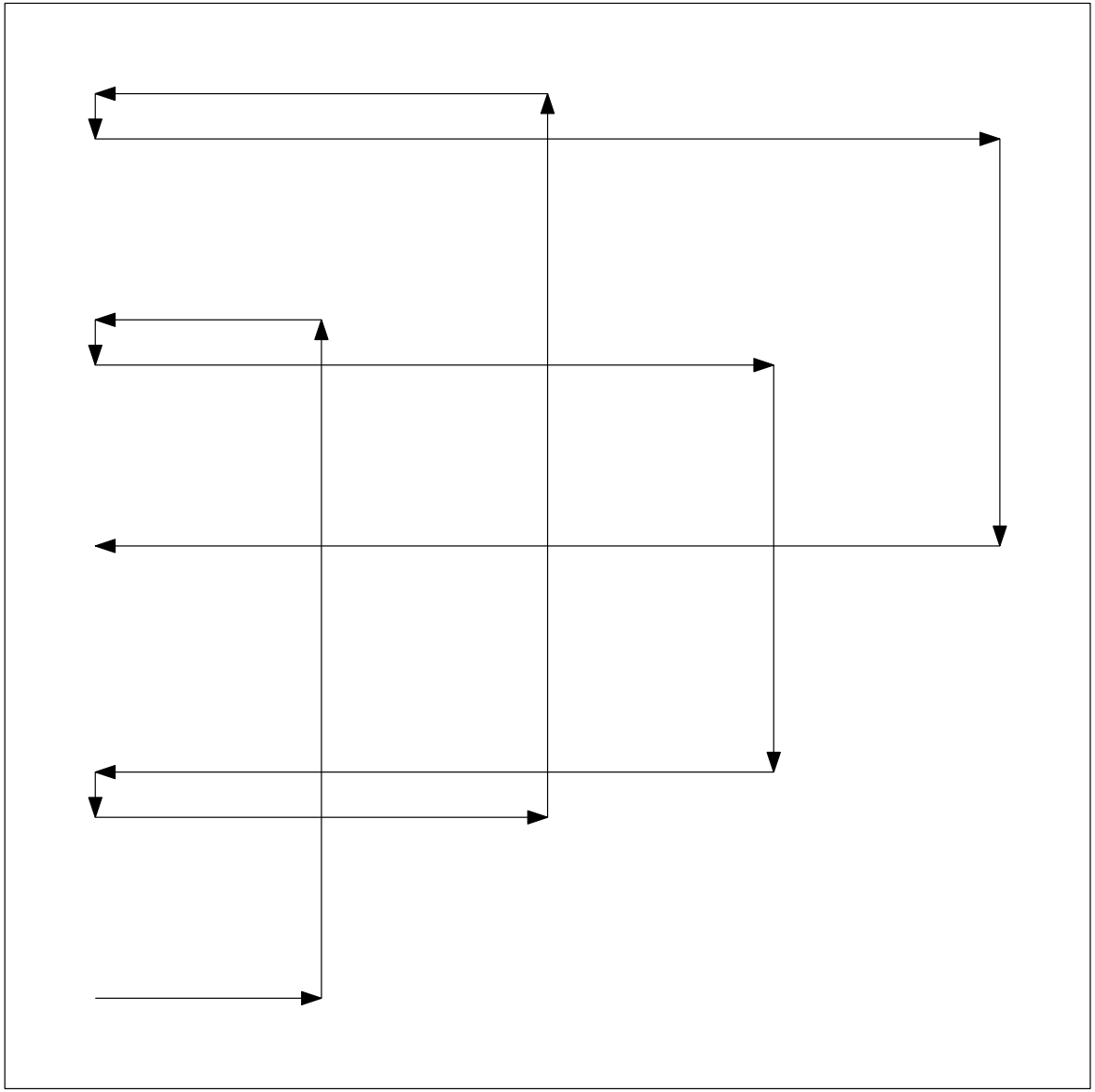

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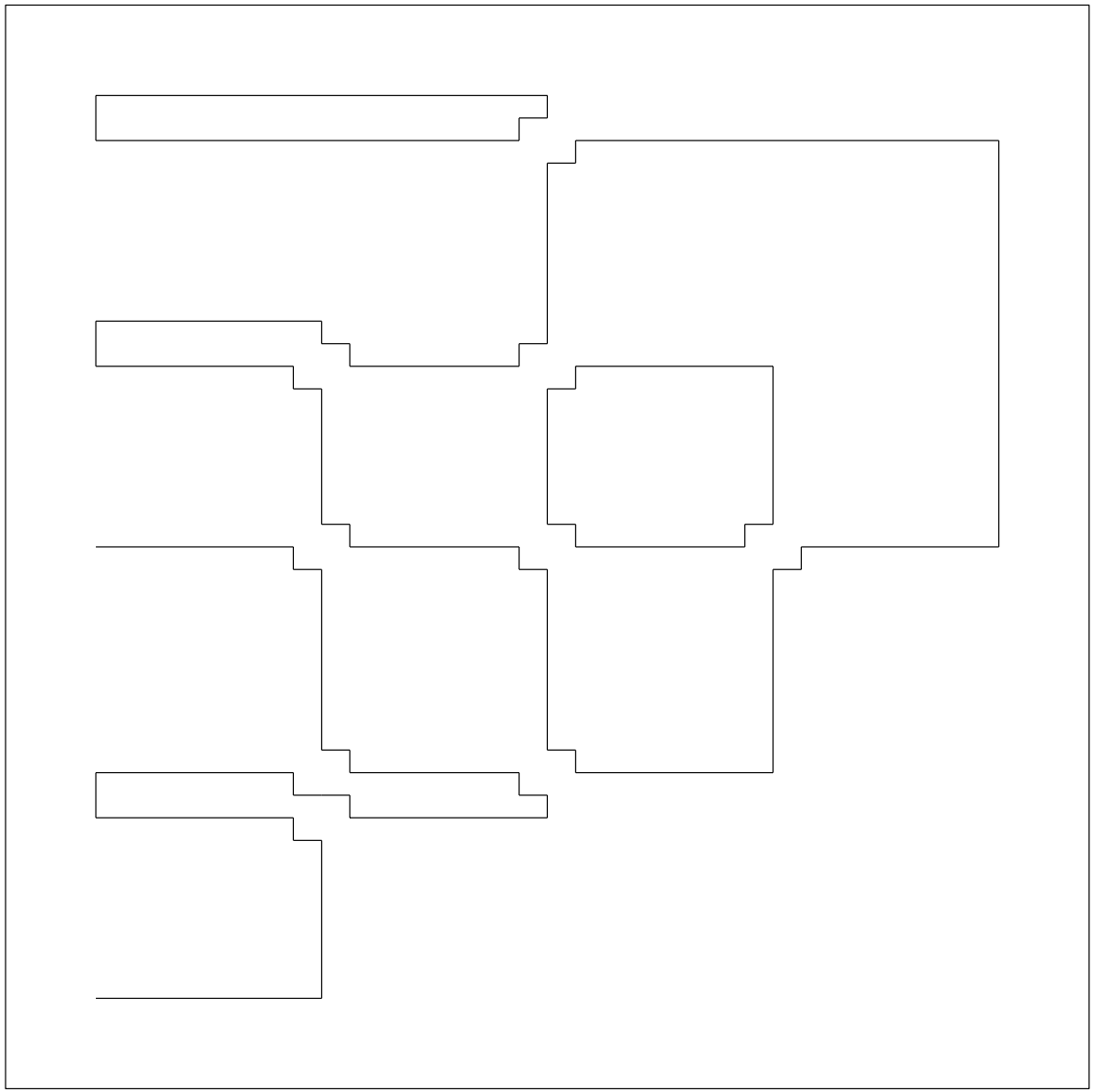

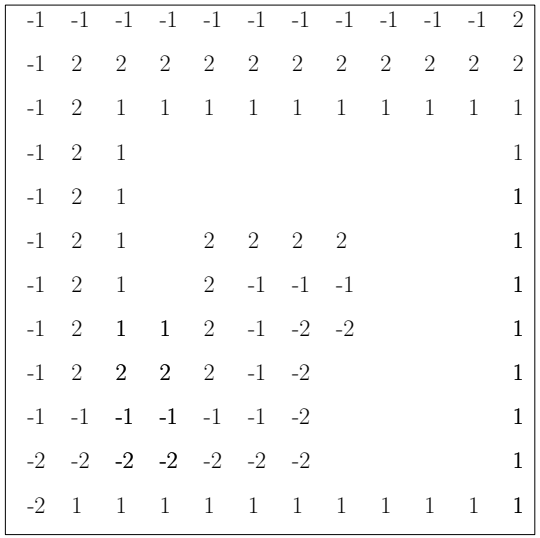

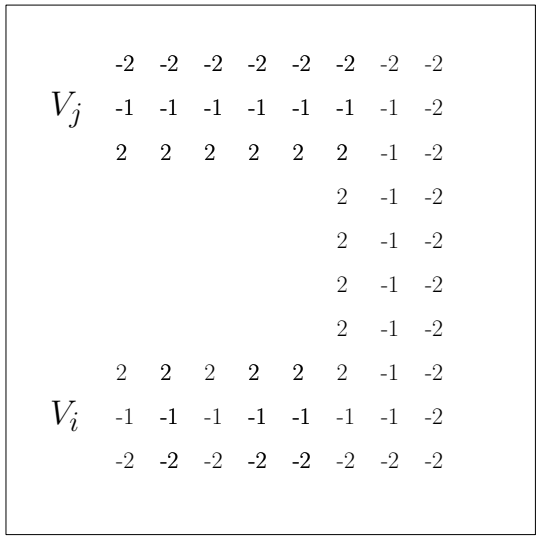

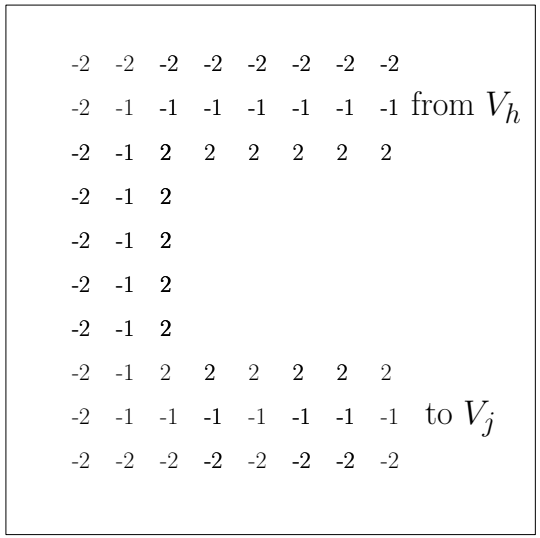

Thank you for your attention!