Decomposability of polygon coverings

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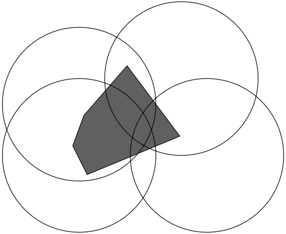

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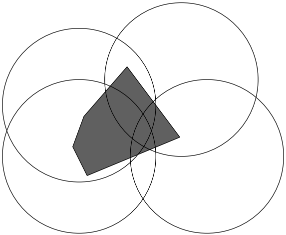

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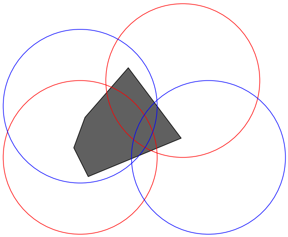

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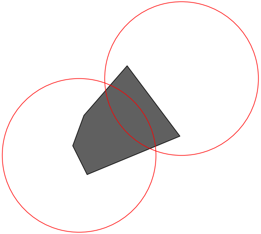

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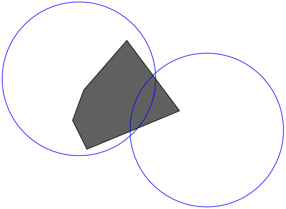

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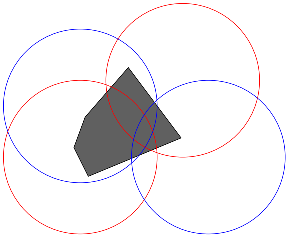

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Deciding plane-cover-decomposability seems harder.

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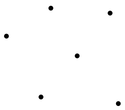

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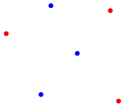

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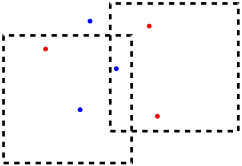

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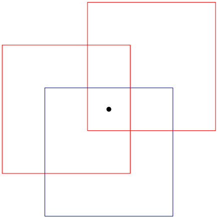

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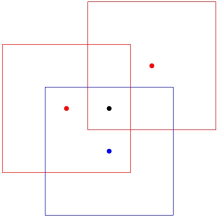

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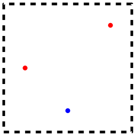

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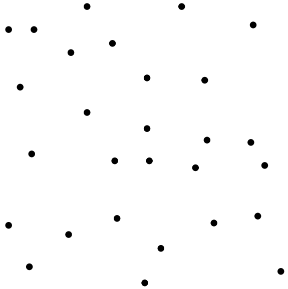

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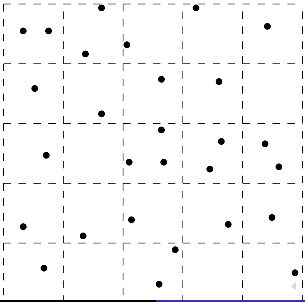

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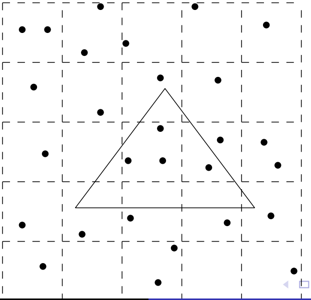

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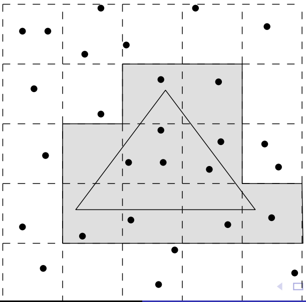

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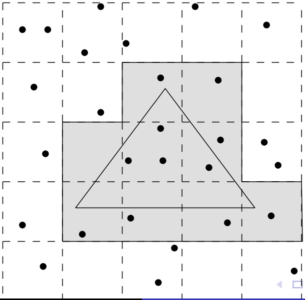

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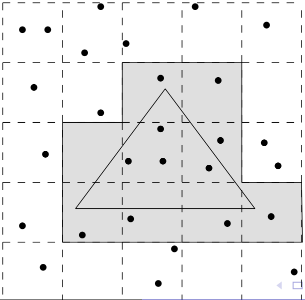

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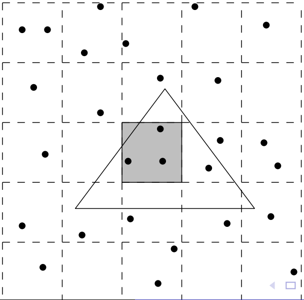

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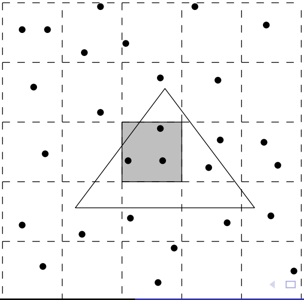
We can assume that all the points are in a small region.



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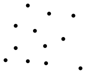

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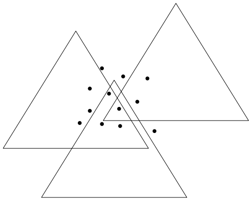

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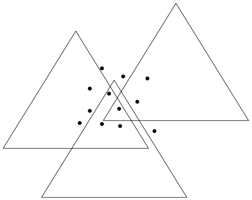

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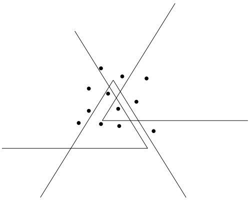

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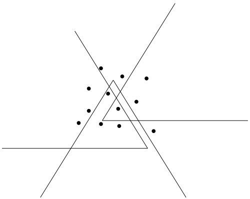

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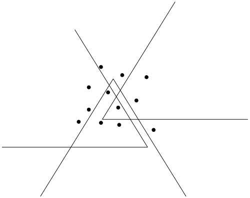
Ie. color points with two colors such that any translate of the given wedges contains both colors if it contains at least k points.



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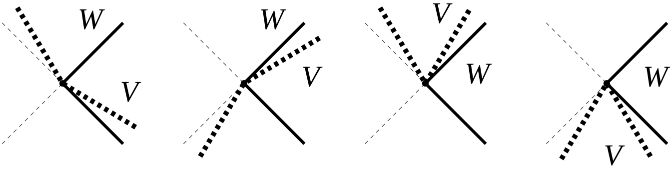

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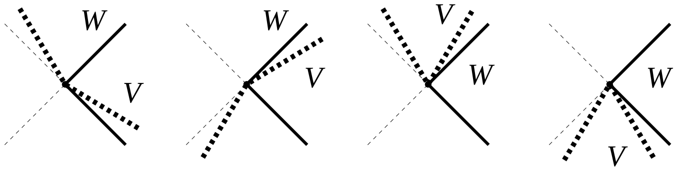
They form a special pair if their relative position is one of these.



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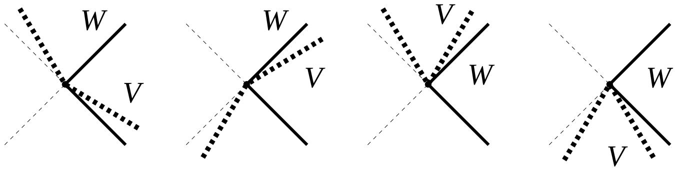
That is, the union of the wedges is in an open halfplane whose boundary contains the origin, but none of them contain the other.



Uncolorable point set for special pair of wedges

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Theorem

For any pair of special wedges V, W and for every k, l, there is a point set of cardinality $\binom{k+l}{k}$ such that for every coloring of P with red and blue either there is a translate of V containing k red and no blue points or there is a translate of W containing l blue and no red points.

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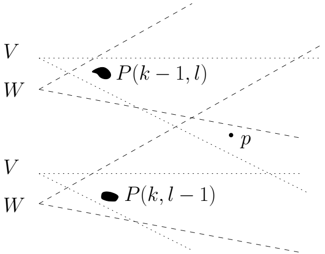

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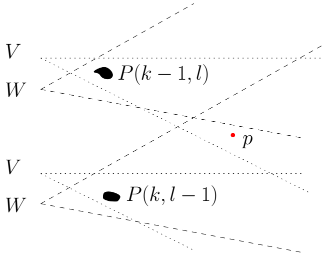

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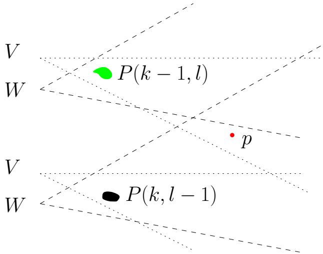

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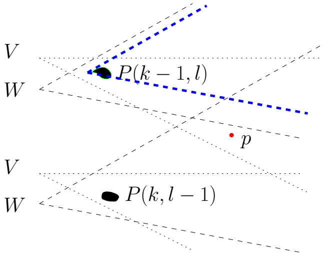

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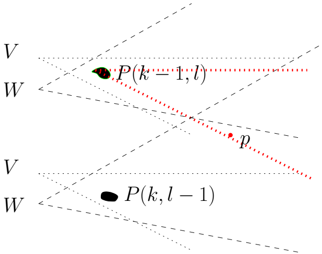

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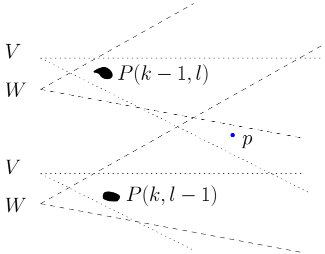

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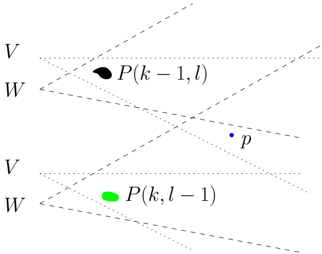

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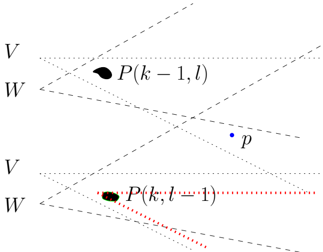

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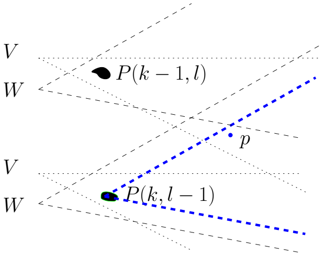

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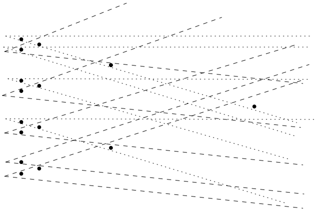

Theorem

If a concave polygon has no parallel sides, then it has a special pair of wedges.

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For a system of wedges we can color the points with two colors such that any translate of the given wedges contains both colors if it contains at least k points IFF none of the pairs are special.

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A polygon is cover-decomposable IFF none of the pair of its angles are special.

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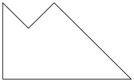

Figure: Cover-decomposable but not union of translates of a convex set.

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For a system of wedges we can color the points with two colors such that any translate of the given wedges contains both colors if it contains at least k points IFF none of the pairs are special.

Theorem

A polygon is cover-decomposable IFF none of the pair of its angles are special.

Corollary

Convex polygons are cover-decomposable.

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A polygon is cover-decomposable IFF none of the pair of its angles are special.

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Convex polygons are cover-decomposable.

Theorem

Polyhedrons are not cover-decomposable.

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Theorem

For a system of wedges we can color the points with two colors such that any translate of the given wedges contains both colors if it contains at least k points IFF none of the pairs are special.

Theorem

A polygon is cover-decomposable IFF none of the pair of its angles are special.

Corollary

Convex polygons are cover-decomposable.

Theorem

Polyhedrons are not cover-decomposable.
Not even space-cover-decomposable.

Different cover-decomposability notions

Different cover-decomposability notions

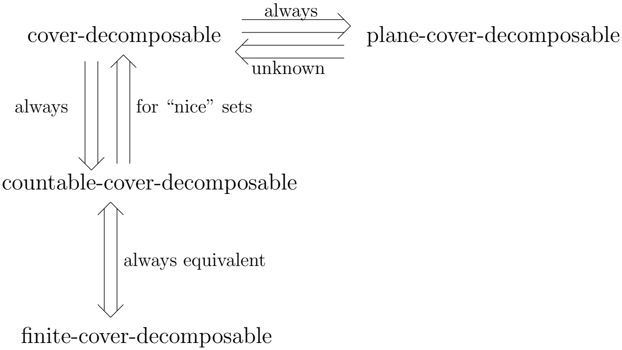


Figure: Connections between variants of cover-decomposability.

Different cover-decomposability notions

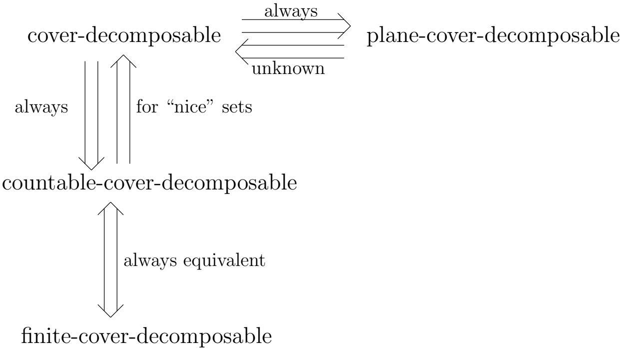


Figure: Connections between variants of cover-decomposability.

Corollary

It does not matter if the set is open or closed.

Extending the counter example to the whole plane

Extending the counter example to the whole plane

Theorem

If a concave polygon C has two special wedges that have a common touching line, then it is not plane-cover-decomposable.

Extending the counter example to the whole plane

Theorem

If a concave polygon C has two special wedges that have a common touching line, then it is not plane-cover-decomposable.

Proof.

The counter example can be extended. □

Extending the counter example to the whole plane

Theorem

If a concave polygon C has two special wedges that have a common touching line, then it is not plane-cover-decomposable.

Proof.

The counter example can be extended. □

Corollary

If a concave pentagon is not cover-decomposable, then it is not plane-cover-decomposable.

Extending the counter example to the whole plane

Theorem

If a concave polygon C has two special wedges that have a common touching line, then it is not plane-cover-decomposable.

Proof.

The counter example can be extended. □

Corollary

If a concave pentagon is not cover-decomposable, then it is not plane-cover-decomposable.

This statement does not hold for hexagons.

Extending the counter example to the whole plane

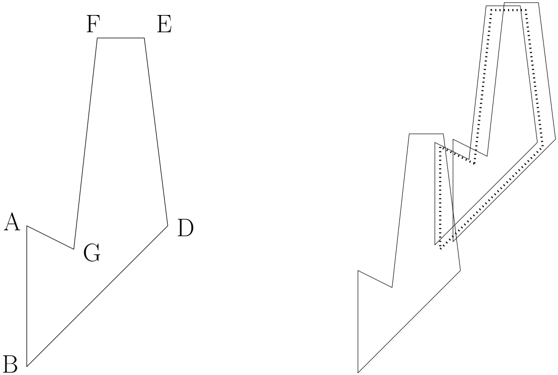
Theorem

There is a hexagon that is not cover-decomposable but it is plane-cover-decomposable.

Extending the counter example to the whole plane

Theorem

There is a hexagon that is not cover-decomposable but it is plane-cover-decomposable.



Thank you for your attention!