An abstract approach to polychromatic coloring: shallow hitting sets in ABA-free hypergraphs and pseudohalfplanes

Balázs Keszegh and Dömötör Pálvölgyi

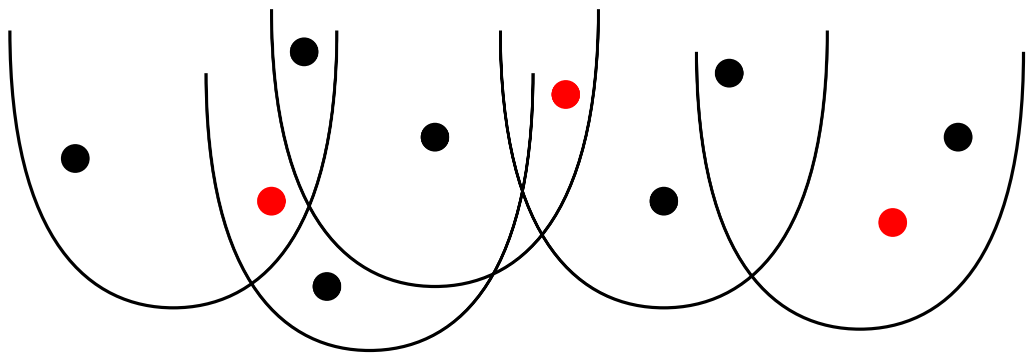
Rényi Institute Budapest   University of Cambridge

DGD 2016, UTE Budapest

Introduction - polychromatic coloring

Problem (Pach, 1980)

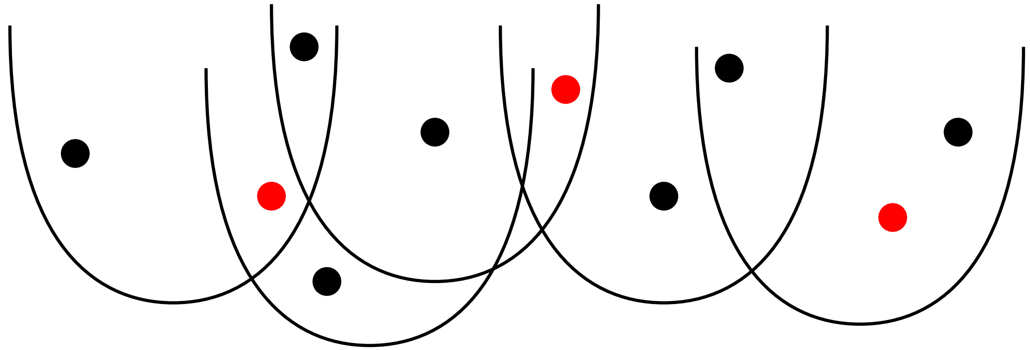
Color points in the plane with k colors such that every *translate of a fixed convex set* containing at least ck points contains all colors.



Introduction - polychromatic coloring

Problem (Pach, 1980)

Color points in the plane with k colors such that every *translate of a fixed convex set* containing at least ck points contains all colors.



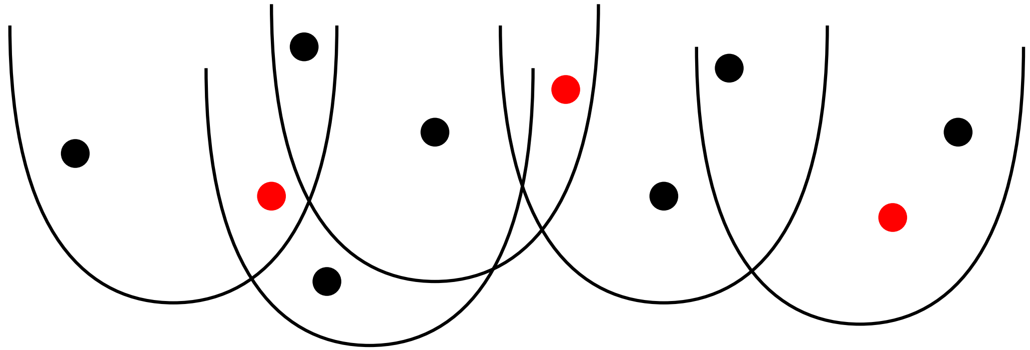
Theorem (Gibson-Varadarajan, 2010)

There is such a c for polygons.

Introduction - polychromatic coloring

Problem (Pach, 1980)

Color points in the plane with k colors such that every *translate of a fixed convex set* containing at least ck points contains all colors.



Theorem (Gibson-Varadarajan, 2010)

There is such a c for polygons.

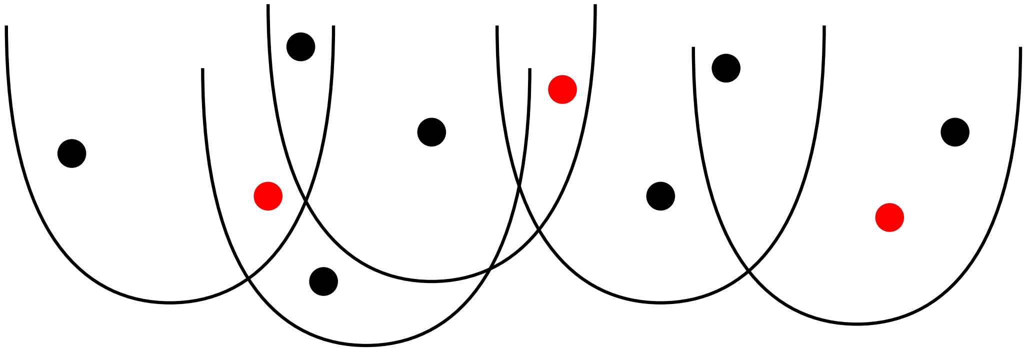
Theorem (P, 2013)

There is no such c for disk,
nor for any other bounded shape with smooth boundary.

Introduction - polychromatic coloring

Problem (Pach, 1980)

Color points in the plane with k colors such that every *translate of a fixed convex set* containing at least ck points contains all colors.



Theorem (Gibson-Varadarajan, 2010)

There is such a c for polygons.

Theorem (P, 2013)

There is no such c for disk,
nor for any other bounded shape with smooth boundary.

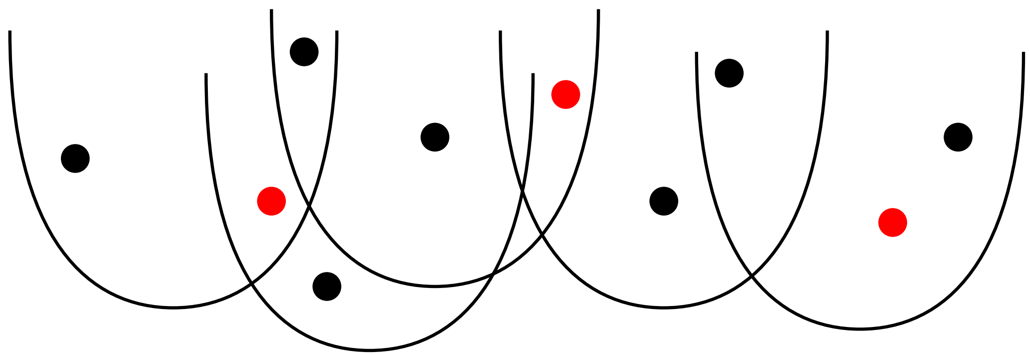
Problem

If c exists for $k = 2$, does c' exist for every k?

Unbounded shapes

Problem

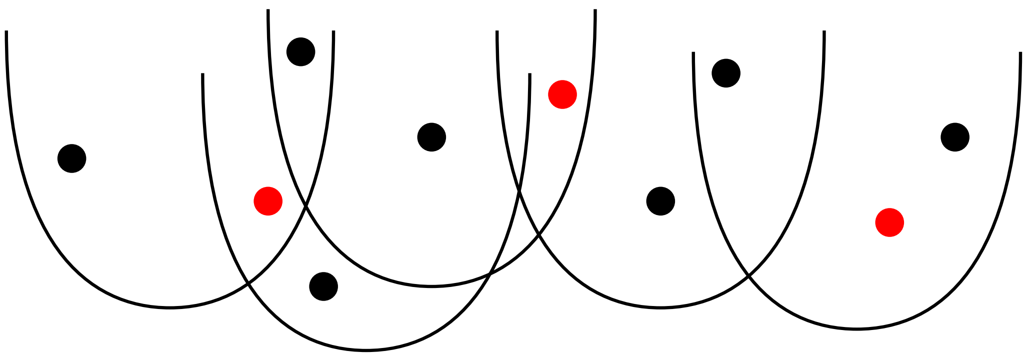
Color points in the plane with k colors such that every *translate of an unbounded convex set* with at least ck points contains all colors.



Unbounded shapes

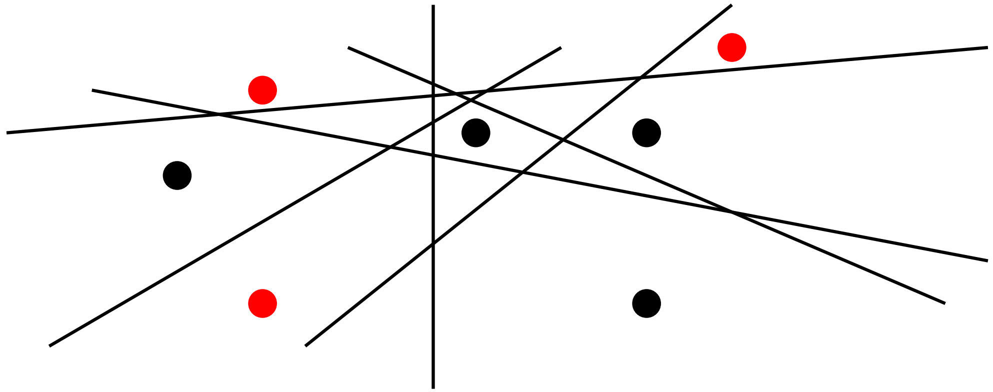
Problem

Color points in the plane with k colors such that every *translate of an unbounded convex set* with at least ck points contains all colors.



Theorem (Smorodinsky and Yuditsky, 2012)

We can color points in the plane with k colors such that every *halfplane* with at least $2k-1$ points contains all colors.



Common generalization: pseudohalfplanes

Problem

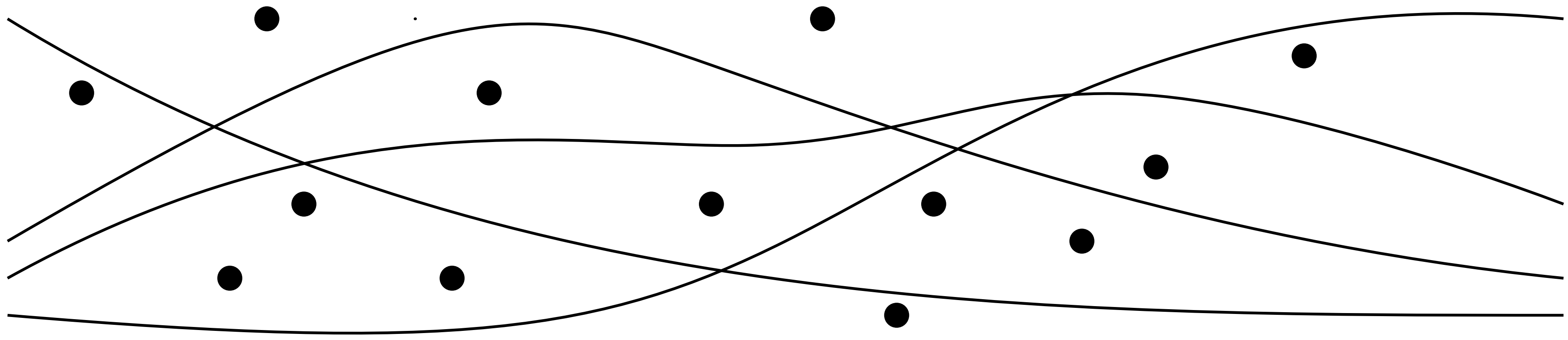
Color points in the plane with k colors such that every *pseudohalfplane* with at least ck points contains all colors.

Common generalization: pseudohalfplanes

Problem

Color points in the plane with k colors such that every *pseudohalfplane* with at least ck points contains all colors.

Remark: we can and will always suppose that the bordering pseudolines are x-monotone.

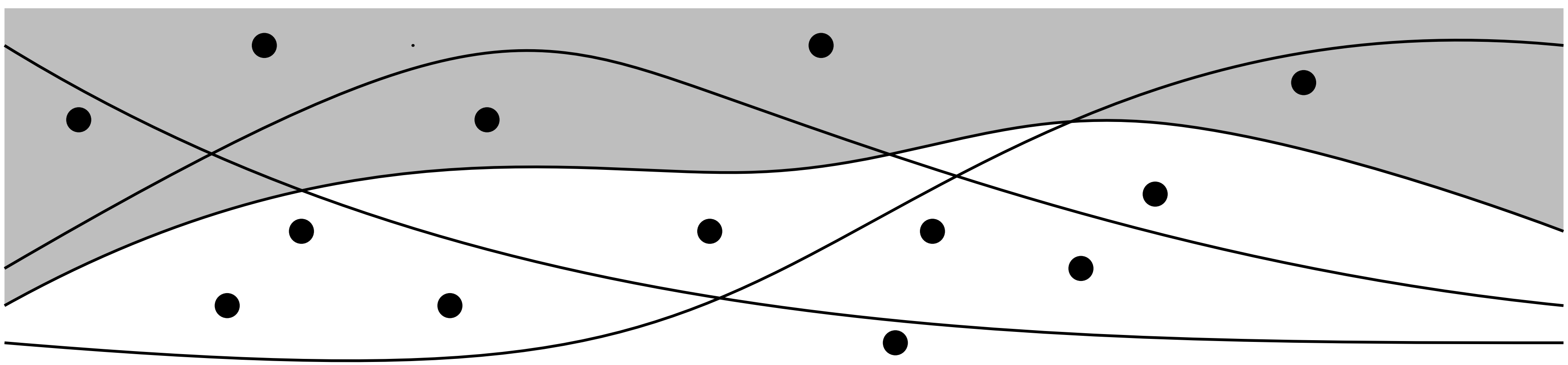


Common generalization: pseudohalfplanes

Problem

Color points in the plane with k colors such that every *pseudohalfplane* with at least ck points contains all colors.

Remark: we can and will always suppose that the bordering pseudolines are x-monotone.



Special case (includes the parabolas but not the halfplanes)

Color points in the plane with k colors such that every *upwards* pseudohalfplane with at least ck points contains all colors.

Abstraction: ABA-free hypergraphs

Definition

A (finite) hypergraph with an ordered vertex set is called *ABA-free* if it does not contain two hyperedges A and B for which there are three vertices $x < y < z$ such that $x, z \in A \setminus B$ and $y \in B \setminus A$.

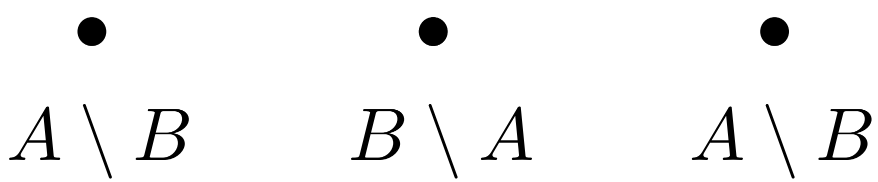
$$\bullet \qquad\qquad \bullet \qquad\qquad \bullet$$

$$A \setminus B \qquad B \setminus A \qquad A \setminus B$$

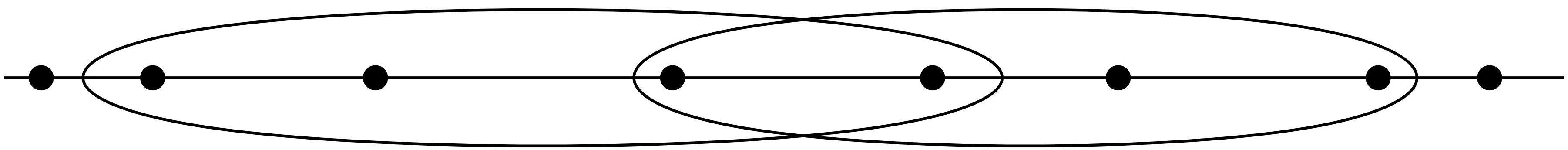
Abstraction: ABA-free hypergraphs

Definition

A (finite) hypergraph with an ordered vertex set is called *ABA-free* if it does not contain two hyperedges A and B for which there are three vertices $x < y < z$ such that $x, z \in A \setminus B$ and $y \in B \setminus A$.



Example: An *interval hypergraph* is a hypergraph whose vertices are some points of the real line and hyperedges are intervals, with the incidences preserved.



Abstraction: ABA-free hypergraphs

Definition

A (finite) hypergraph with an ordered vertex set is called *ABA-free* if it does not contain two hyperedges A and B for which there are three vertices $x < y < z$ such that $x, z \in A \setminus B$ and $y \in B \setminus A$.

Claim

Ordering the points by their x-coordinates, upwards pseudohalfplanes define an ABA-free hypergraph.

Abstraction: ABA-free hypergraphs

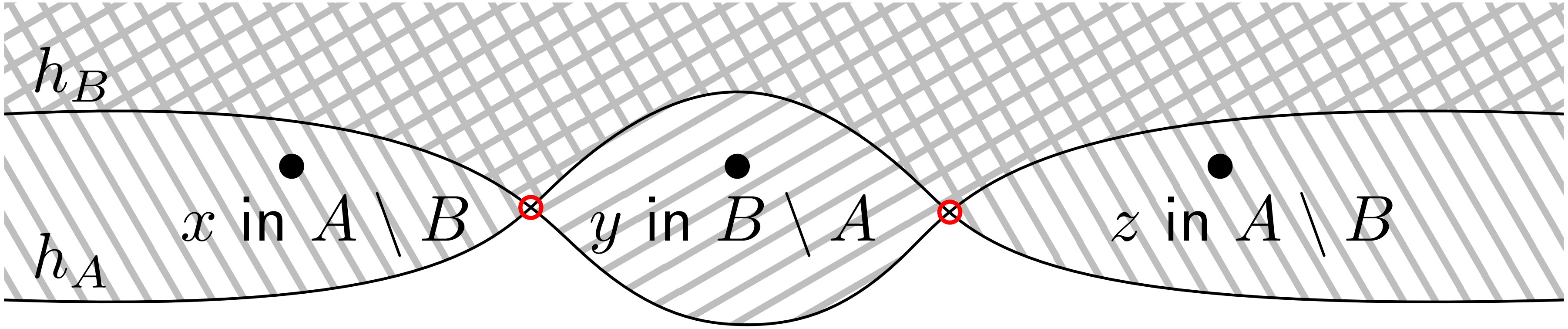
Definition

A (finite) hypergraph with an ordered vertex set is called *ABA-free* if it does not contain two hyperedges A and B for which there are three vertices $x < y < z$ such that $x, z \in A \setminus B$ and $y \in B \setminus A$.

Claim

Ordering the points by their x-coordinates, upwards pseudohalfplanes define an ABA-free hypergraph.

Proof No ABA-occurrence, otherwise the corresponding pseudohalfplanes must intersect twice, a contradiction.



Abstraction: ABA-free hypergraphs

Definition

A (finite) hypergraph with an ordered vertex set is called *ABA-free* if it does not contain two hyperedges A and B for which there are three vertices $x < y < z$ such that $x, z \in A \setminus B$ and $y \in B \setminus A$.

Claim

Ordering the points by their x-coordinates, upwards pseudohalfplanes define an ABA-free hypergraph.

Claim

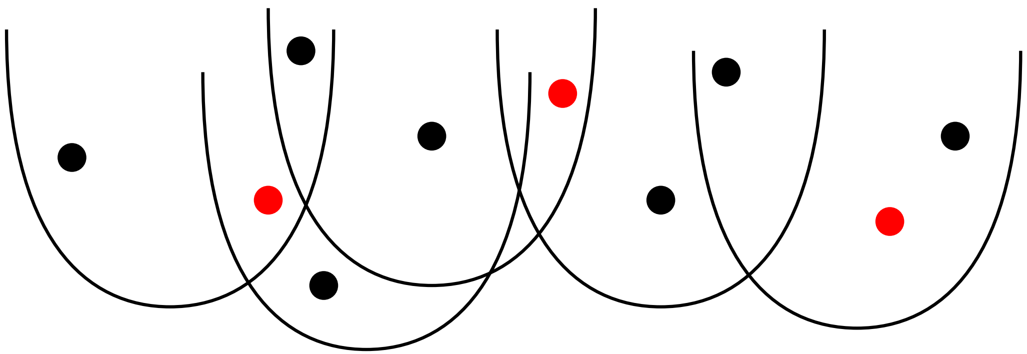
The opposite is also true: for every ordered ABA-free hypergraph there is a point set and an upwards pseudohalfplane arrangement whose incidences give it.

Corollary We can imagine ABA-free hypergraphs as upwards pseudohalfplanes.

Polychromatic coloring ABA-free hypergraphs

Theorem

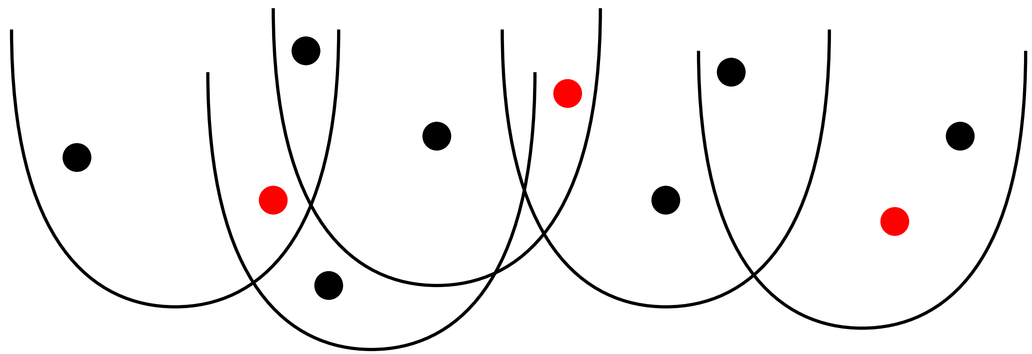
Given an ABA-free $\mathcal{H}$ we can color its vertices with k colors such that every $A \in \mathcal{H}$ of size $2k-1$ has all k colors.



Polychromatic coloring ABA-free hypergraphs

Theorem

Given an ABA-free $\mathcal{H}$ we can color its vertices with k colors such that every $A \in \mathcal{H}$ of size $2k-1$ has all k colors.

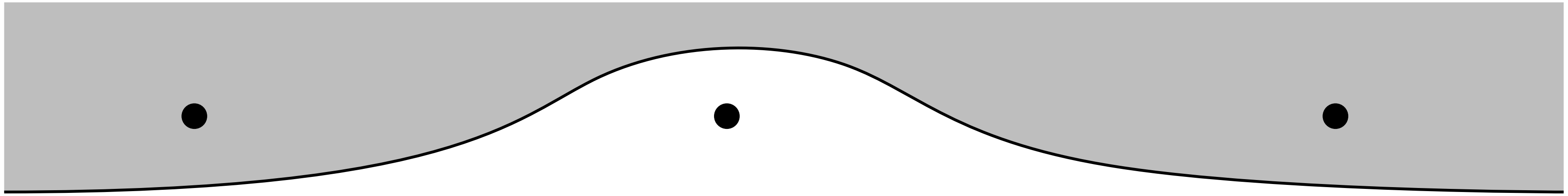


Proof

We modify the proof of Smorodinsky and Yuditsky for halfplanes step by step for abstraction.

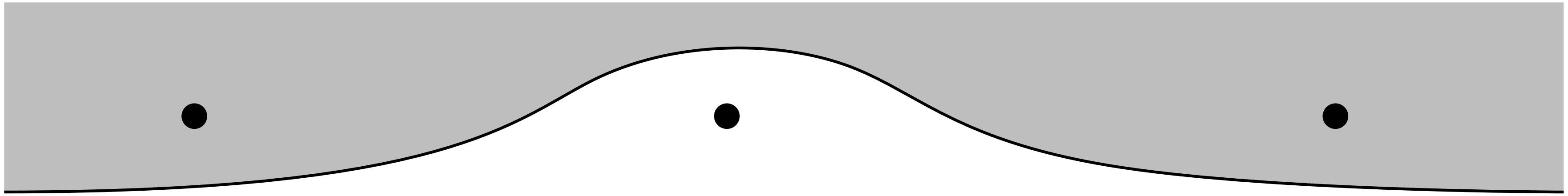
Proof for ABA-free hypergraphs

A vertex a is *skippable* if there exists an $A \in \mathcal{H}$ such that $\min(A) < a < \max(A)$ and $a \notin A$.
A vertex a is *unskippable* if there is no such A.



Proof for ABA-free hypergraphs

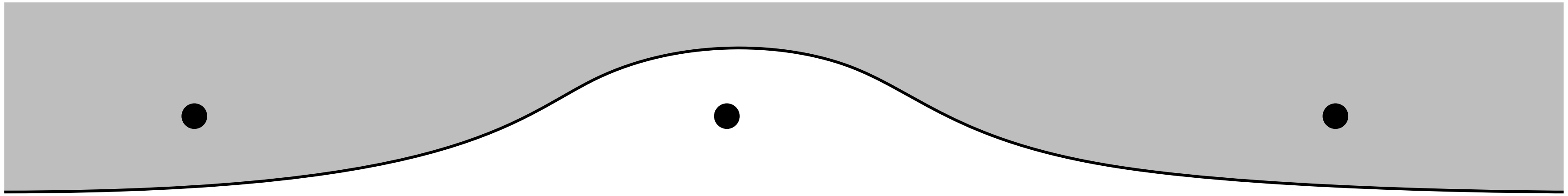
A vertex a is *skippable* if there exists an $A \in \mathcal{H}$ such that $\min(A) < a < \max(A)$ and $a \notin A$.
A vertex a is *unskippable* if there is no such A.



Lemma 1 If $\mathcal{H}$ is ABA-free, then every $A \in \mathcal{H}$ contains an unskippable vertex.

Proof for ABA-free hypergraphs

A vertex a is *skippable* if there exists an $A \in \mathcal{H}$ such that $\min(A) < a < \max(A)$ and $a \notin A$.
A vertex a is *unskippable* if there is no such A.

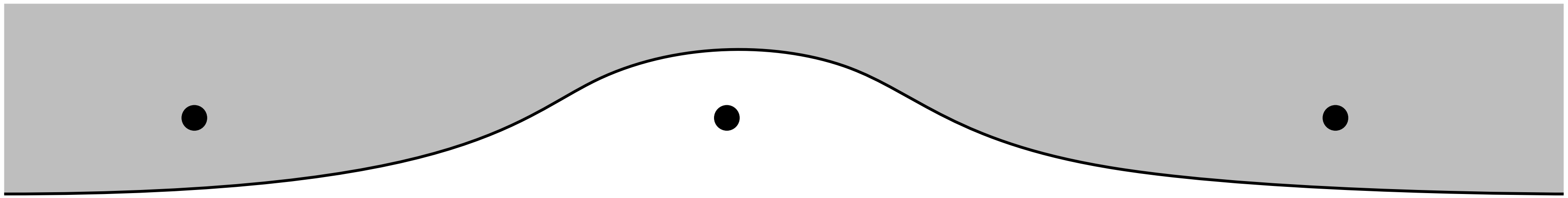


Lemma 1 If $\mathcal{H}$ is ABA-free, then every $A \in \mathcal{H}$ contains an unskippable vertex.

Corollary Unskippable vertices form a hitting set.

Proof for ABA-free hypergraphs

A vertex a is *skippable* if there exists an $A \in \mathcal{H}$ such that $\min(A) < a < \max(A)$ and $a \notin A$.
A vertex a is *unskippable* if there is no such A.



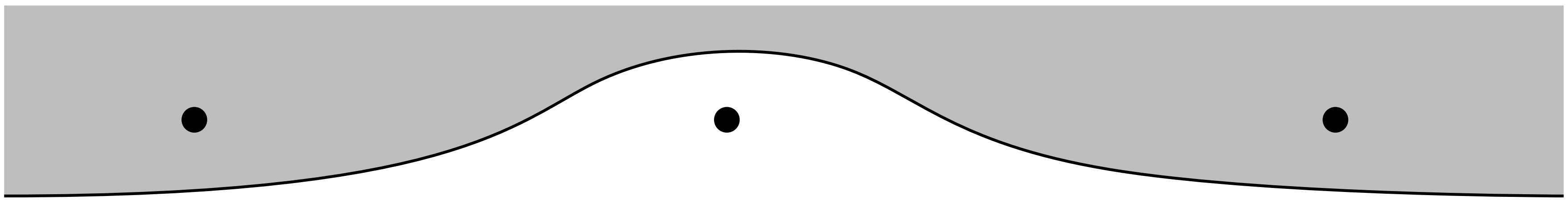
Lemma 1 If $\mathcal{H}$ is ABA-free, then every $A \in \mathcal{H}$ contains an unskippable vertex.

Corollary Unskippable vertices form a hitting set.

Lemma 2 If $\mathcal{H}$ is ABA-free and containment-free, then any minimal hitting set containing only unskippables is 2-shallow.

Proof for ABA-free hypergraphs

A vertex a is *skippable* if there exists an $A \in \mathcal{H}$ such that $\min(A) < a < \max(A)$ and $a \notin A$.
A vertex a is *unskippable* if there is no such A.



Lemma 1 If $\mathcal{H}$ is ABA-free, then every $A \in \mathcal{H}$ contains an unskippable vertex.

Corollary Unskippable vertices form a hitting set.

Lemma 2 If $\mathcal{H}$ is ABA-free and containment-free, then any minimal hitting set containing only unskippables is 2-shallow.

Definition
S is a c*-shallow hitting set* if $1 \leq |S \cap H| \leq c$ for every $H \in \mathcal{H}$.

Proof for ABA-free hypergraphs

Theorem

Given an ABA-free $\mathcal{H}$ in which each hyperedge has size at least $2k-1$, we can color its vertices with k colors such that every hyperedge contains all k colors.

Proof for ABA-free hypergraphs

Theorem

Given an ABA-free $\mathcal{H}$ in which each hyperedge has size at least $2k-1$, we can color its vertices with k colors such that every hyperedge contains all k colors.

Proof

Until there are two edges of $\mathcal{H}$ one containing the other, delete the bigger one to make $\mathcal{H}$ containment-free.

Proof for ABA-free hypergraphs

Theorem

Given an ABA-free $\mathcal{H}$ in which each hyperedge has size at least $2k-1$, we can color its vertices with k colors such that every hyperedge contains all k colors.

Proof

Until there are two edges of $\mathcal{H}$ one containing the other, delete the bigger one to make $\mathcal{H}$ containment-free.

Using lemmas, minimal hitting set S containing only unskippables is 2-shallow.

Proof for ABA-free hypergraphs

Theorem

Given an ABA-free $\mathcal{H}$ in which each hyperedge has size at least $2k-1$, we can color its vertices with k colors such that every hyperedge contains all k colors.

Proof

Until there are two edges of $\mathcal{H}$ one containing the other, delete the bigger one to make $\mathcal{H}$ containment-free.

Using lemmas, minimal hitting set S containing only unskippables is 2-shallow.

Color S with first color, then delete S.

Proof for ABA-free hypergraphs

Theorem

Given an ABA-free $\mathcal{H}$ in which each hyperedge has size at least $2k-1$, we can color its vertices with k colors such that every hyperedge contains all k colors.

Proof

Until there are two edges of $\mathcal{H}$ one containing the other, delete the bigger one to make $\mathcal{H}$ containment-free.

Using lemmas, minimal hitting set S containing only unskippables is 2-shallow.

Color S with first color, then delete S.

Every hyperedge still has $2k-3$ vertices, induction. $\blacksquare$

Generalization to pseudohalfplanes

Theorem (pseudohalfplanes)

Given a finite set of points and a pseudohalfplane arrangement, we can color points with k colors such that any pseudohalfplane that has $2k-1$ points contains all k colors.

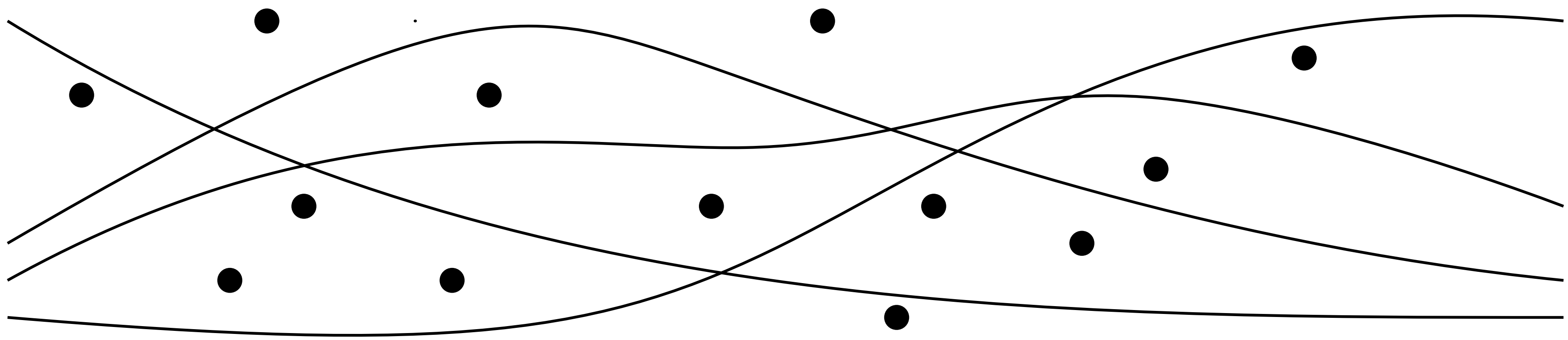
Generalization to pseudohalfplanes

Theorem (pseudohalfplanes)

Given a finite set of points and a pseudohalfplane arrangement, we can color points with k colors such that any pseudohalfplane that has $2k-1$ points contains all k colors.

Proof is based on

A general pseudohalfplane arrangement is the union of an upwards pseudohalfplane arrangement $\mathcal{F}$ (which is an ABA-free hypergraph) and the family $\bar{\mathcal{F}}$ of the complements of the hyperedges in $\mathcal{F}$ (which is also ABA-free).



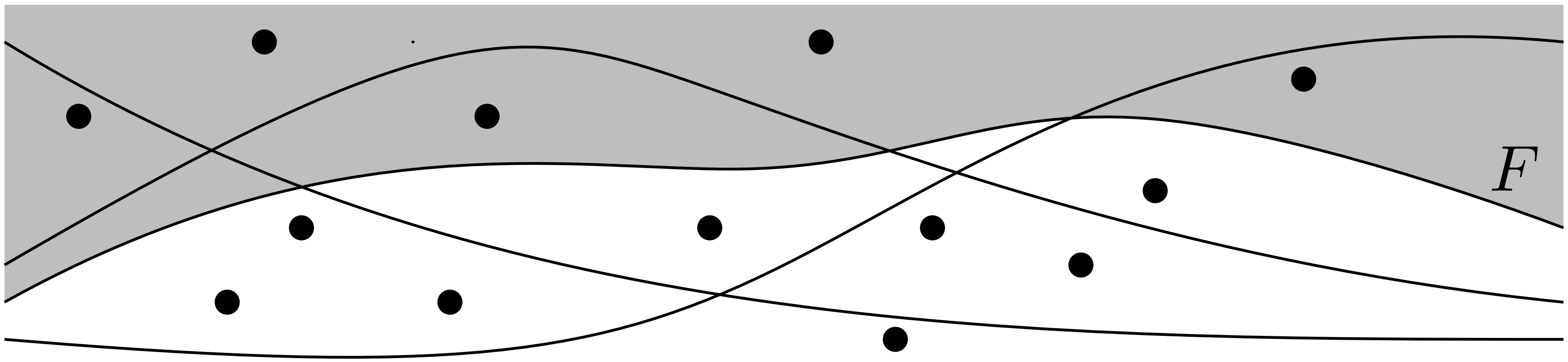
Generalization to pseudohalfplanes

Theorem (pseudohalfplanes)

Given a finite set of points and a pseudohalfplane arrangement, we can color points with k colors such that any pseudohalfplane that has $2k-1$ points contains all k colors.

Proof is based on

A general pseudohalfplane arrangement is the union of an upwards pseudohalfplane arrangement $\mathcal{F}$ (which is an ABA-free hypergraph) and the family $\bar{\mathcal{F}}$ of the complements of the hyperedges in $\mathcal{F}$ (which is also ABA-free).



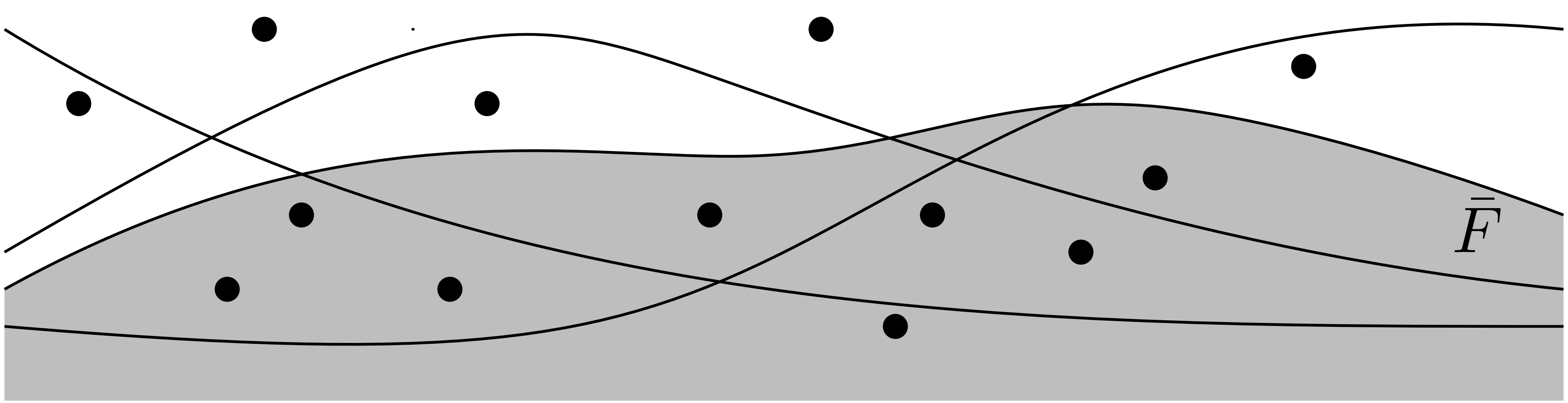
Generalization to pseudohalfplanes

Theorem (pseudohalfplanes)

Given a finite set of points and a pseudohalfplane arrangement, we can color points with k colors such that any pseudohalfplane that has $2k-1$ points contains all k colors.

Proof is based on

A general pseudohalfplane arrangement is the union of an upwards pseudohalfplane arrangement $\mathcal{F}$ (which is an ABA-free hypergraph) and the family $\bar{\mathcal{F}}$ of the complements of the hyperedges in $\mathcal{F}$ (which is also ABA-free).



Dual result for ABA-free hypergraphs

Observation

The dual of an ABA-free hypergraph is also ABA-free with (any extension of the partial) ordering defined by

$$A < B \text{ iff } \exists x < y, x \in A \setminus B, y \in B \setminus A.$$

Dual result for ABA-free hypergraphs

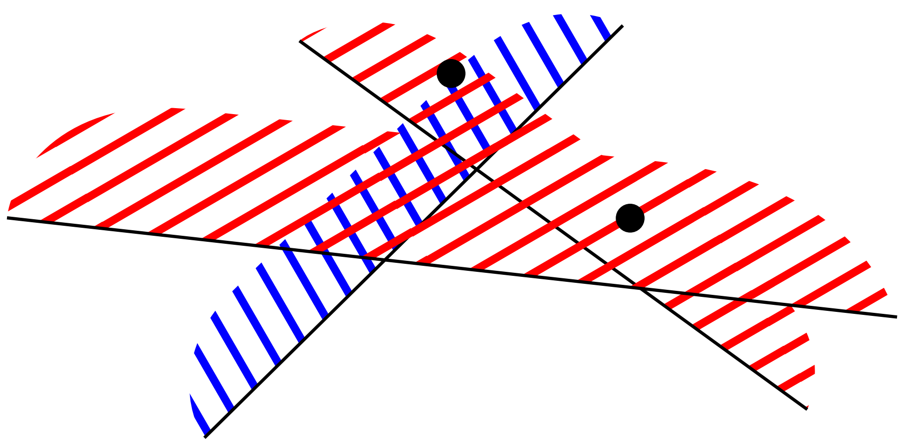
Observation

The dual of an ABA-free hypergraph is also ABA-free with (any extension of the partial) ordering defined by

$$A < B \text{ iff } \exists x < y, x \in A \setminus B, y \in B \setminus A.$$

Corollary

Hyperedges of an ABA-free hypergraph can be colored with k colors such that if a vertex is in $2k-1$ hyperedges, then it is covered by all k color classes.



Generalizations

Theorem (dual for pseudohalfplanes)

Pseudohalfplanes can be colored with k colors such that if a point is in $3k-2$ pseudohalfplanes, then it is covered by all k color classes.

Generalizations

Theorem (dual for pseudohalfplanes)

Pseudohalfplanes can be colored with k colors such that if a point is in $3k-2$ pseudohalfplanes, then it is covered by all k color classes.

Lemma (Helly)

Every containment-free dual pseudohalfplane-hypergraph has a 3-shallow hitting set.

Generalizations

Theorem (dual for pseudohalfplanes)

Pseudohalfplanes can be colored with k colors such that if a point is in $3k-2$ pseudohalfplanes, then it is covered by all k color classes.

Lemma (Helly)

Every containment-free dual pseudohalfplane-hypergraph has a 3-shallow hitting set.

Common generalization of primal and dual

Pseudohalfsphere-hypergraph on S is determined by an ABA-free $\mathcal{F}$ and an $X \subset S$, such that every hyperedge equals $F \Delta X$ or $\bar{F} \Delta X$ for some $F \in \mathcal{F}$.

Generalizations

Theorem (dual for pseudohalfplanes)

Pseudohalfplanes can be colored with k colors such that if a point is in $3k-2$ pseudohalfplanes, then it is covered by all k color classes.

Lemma (Helly)

Every containment-free dual pseudohalfplane-hypergraph has a 3-shallow hitting set.

Common generalization of primal and dual

Pseudohalfsphere-hypergraph on S is determined by an ABA-free $\mathcal{F}$ and an $X \subset S$, such that every hyperedge equals $F \Delta X$ or $\bar{F} \Delta X$ for some $F \in \mathcal{F}$.

Remarks Self-dual. If $X = \emptyset$, we get pseudohalfplanes.

Generalizations

Theorem (dual for pseudohalfplanes)

Pseudohalfplanes can be colored with k colors such that if a point is in $3k-2$ pseudohalfplanes, then it is covered by all k color classes.

Lemma (Helly)

Every containment-free dual pseudohalfplane-hypergraph has a 3-shallow hitting set.

Common generalization of primal and dual

Pseudohalfsphere-hypergraph on S is determined by an ABA-free $\mathcal{F}$ and an $X \subset S$, such that every hyperedge equals $F \Delta X$ or $\bar{F} \Delta X$ for some $F \in \mathcal{F}$.

Remarks Self-dual. If $X = \emptyset$, we get pseudohalfplanes.

Theorem

Helly gives 4-shallow for pseudohalfspheres, from that $4k-3$.

Summary of results

	Polychrom. k-col.	Shallow hitting set
ABA-free hypergraphs	$2k-1$	2
Pseudohalfplanes	$2k-1$	2
Dual pseudohalfplanes	$\leq 3k-2$	≤ 3
Pseudohalfspheres	$\leq 4k-3$	≤ 4

Summary of results

	Polychrom. k-col.	Shallow hitting set
ABA-free hypergraphs	$2k-1$	2
Pseudohalfplanes	$2k-1$	2
Dual pseudohalfplanes	$\leq 3k-2$	≤ 3
Pseudohalfspheres	$\leq 4k-3$	≤ 4

Remark For dual halfplanes the answer is 3 for $k=2$ (Fulek)

Summary of results

	Polychrom. k-col.	Shallow hitting set
ABA-free hypergraphs	$2k-1$	2
Pseudohalfplanes	$2k-1$	2
Dual pseudohalfplanes	$\leq 3k-2$	≤ 3
Pseudohalfspheres	$\leq 4k-3$	≤ 4

Remark For dual halfplanes the answer is 3 for $k=2$ (Fulek)

For more related, see www.cs.elte.hu/˜dom/covdec

Summary of results

	Polychrom. k-col.	Shallow hitting set
ABA-free hypergraphs	$2k-1$	2
Pseudohalfplanes	$2k-1$	2
Dual pseudohalfplanes	$\leq 3k-2$	≤ 3
Pseudohalfspheres	$\leq 4k-3$	≤ 4

Remark For dual halfplanes the answer is 3 for $k=2$ (Fulek)

For more related, see www.cs.elte.hu/~dom/covdec

Thank you for your attention!

Summary of results

	Polychrom. k-col.	Shallow hitting set
ABA-free hypergraphs	$2k-1$	2
Pseudohalfplanes	$2k-1$	2
Dual pseudohalfplanes	$\leq 3k-2$	≤ 3
Pseudohalfspheres	$\leq 4k-3$	≤ 4

Remark For dual halfplanes the answer is 3 for $k=2$ (Fulek)

For more related, see www.cs.elte.hu/˜dom/covdec

Thank you for your attention!

Közsönöm szepen!