Edge-ordered Graphs: Geometric Applications

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MTA-ELTE CoGe, Budapest

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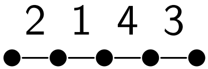

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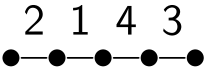

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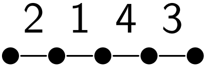

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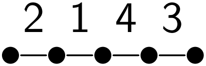
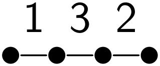


contains


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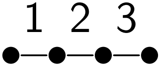
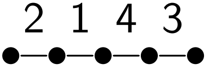
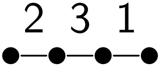


No order on vertices,


also contains


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Classic: $\mathrm{ex}_<(n, \bullet\overset{1}{-}\bullet\overset{2}{-}\bullet \dots \bullet\overset{k}{-}\bullet) = O(n)$

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Systematic study of $\mathrm{ex}_<(n, H)$.

Kaucheriya, Tardos '22+−+:
Characterisation of H with $\mathrm{ex}_<(n, H)$ (almost) linear.

$\mathrm{ex}_<(n, P_5)$

Gerbner, Methuku, Nagy, P., Tardos, Vizer '20+:

Turán numbers of edge-ordered paths with four edges		
Labeling	Lower bound	Upper bound
$\{1234, 4321\}$	$\Omega(n)$	$O(n)$
$\{1243, 3421, 4312, 2134\}$	$\Omega(n)$	$O(n)$
$\{1324, 4231\}$	$\Omega(n \log n)$	$O(n \log n)$
$\{1432, 2341, 4123, 3214\}$	$\Omega(n \log n)$	$O(n \log n)$
$\{2143, 3412\}$	$\Omega(n \log n)$	$O(n \log n)$
$\{1342, 2431, 4213, 3124\}$	$\Omega(n \log n)$	$O(n \log^2 n)$
$\{2413, 3142\}$	$\binom{n}{2}$	$\binom{n}{2}$
$\{1423, 3241, 4132, 2314\}$	$\binom{n}{2}$	$\binom{n}{2}$

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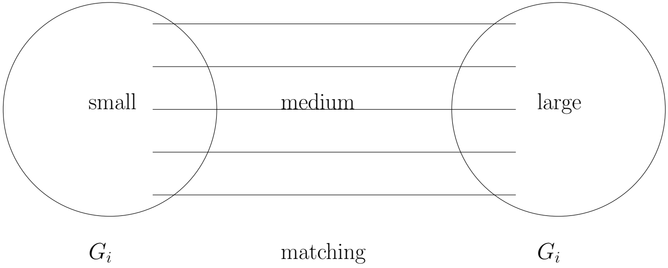
$\mathrm{ex}_<(n, \bullet\overset{2}{-}\bullet\overset{1}{-}\bullet\overset{4}{-}\bullet\overset{3}{-}\bullet) = \Omega(n \log n)$: Construct G_{i+1} from G_i as



$$F(n) = n/2 + 2F(n/2) \Rightarrow F(n) = \Theta(n \log n)$$

Application: Unit distances in convex position

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We reprove this with edge-ordered graphs.

$O(n \log n)$ unit distances in convex position

Keep only almost horizontal edges whose left and right endpoints are closer than 0.1.

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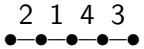
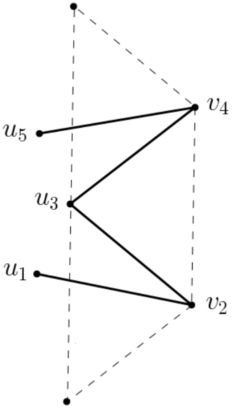
This graph has no 

because middle vertex u_3 would be in convex hull of other four.



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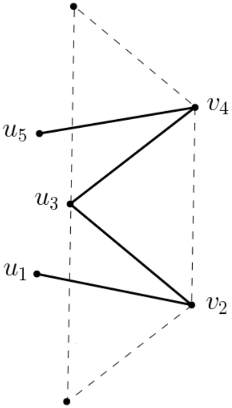
$\mathrm{ex}_<(n, \bullet\overset{2}{-}\bullet\overset{1}{-}\bullet\overset{4}{-}\bullet\overset{3}{-}\bullet) = O(n \log n)$
$\Rightarrow$ same bound for unit distances in convex position.



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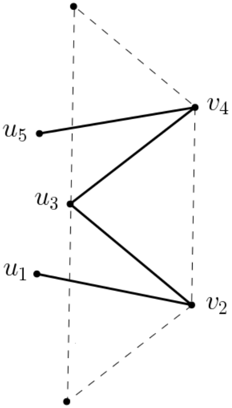
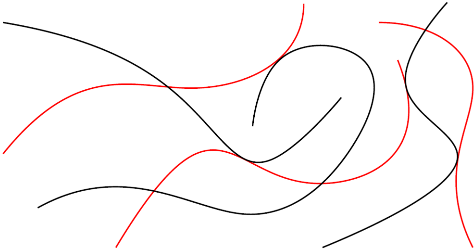
Open: Can we prove a linear bound using other edge-order?



Application: Tangencies among curves

Pinchasi and Ben-Dan: What is the maximum number of tangencies between the members of two families, each of which consists of pairwise disjoint curves?



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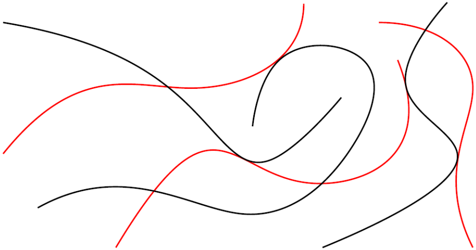

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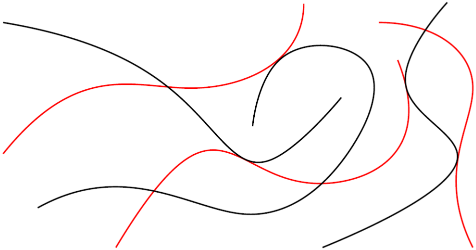

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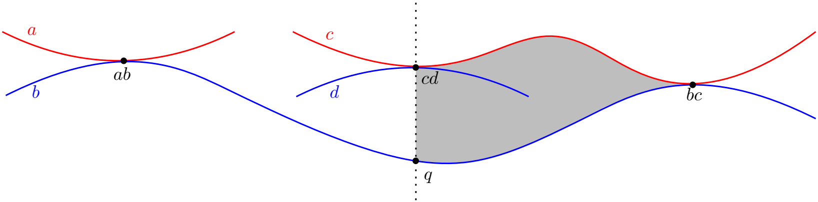
Keszegh and P. '21+ improved this to optimal $O(n \log n)$.

$O(n \log n)$ tangencies among two x-monotone families

Take bipartite graph of tangencies with edges ordered according to their x-coordinates, so $ab < cd < bc$.



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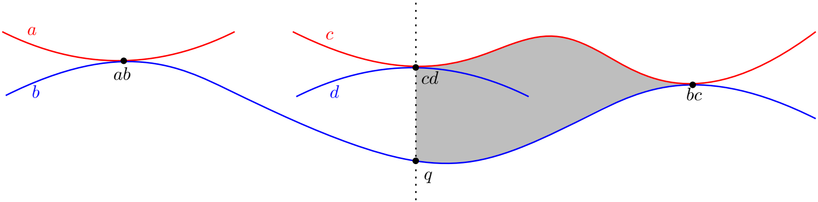
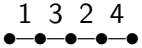


This graph has no 

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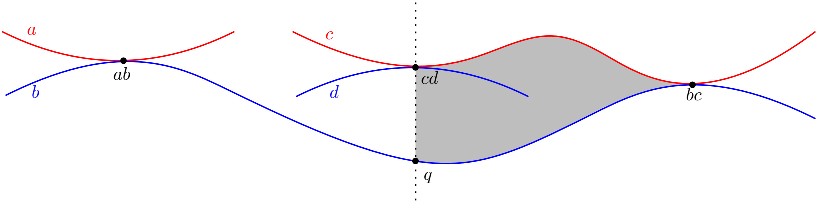

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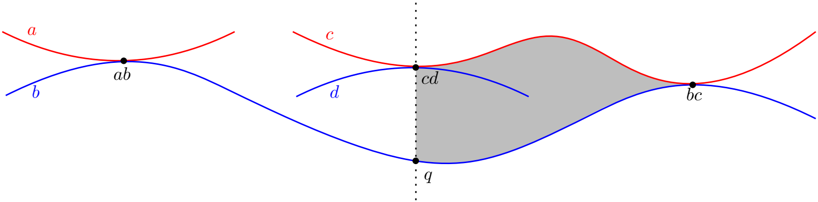
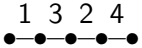


This graph has no ; suppose $ab < cd < bc < de$ ↯

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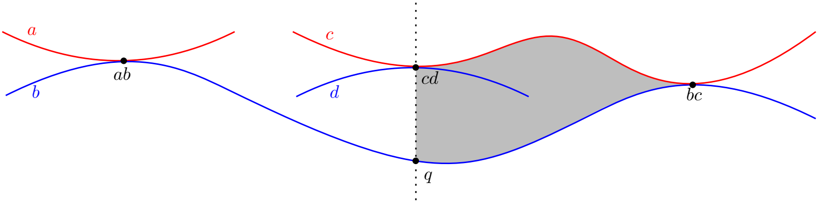

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Tangencies among two disjoint families

Open: What is best bound without x-monotonity?

Curve type	# of tangencies	Lower bd ref	Upper bd ref
general	$\Omega(n^{4/3})$, $\tilde{O}(n^{3/2})$	Keszegh & P.	Pinchasi & Ben-Dan
doubly-grounded	Keszegh & P. $\Rightarrow^*$ $\Theta(n^{4/3})$	Keszegh & P.	Keszegh & P.
x-monotone	$\Theta(n \log n)$	Pach, Suk & Treml	Keszegh & P.
$\leq$ 1-intersecting	$\Theta(n)$	trivial	Ackerman, Keszegh & P.
convex regions	$\Theta(n)$	trivial	Pach, Suk & Treml

* Assuming a conjecture of Pach and Tardos on 0-1 matrices.

Thank you for your attention!