Exchange properties of finite set-systems

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Joint work with
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"No question, this is high quality combinatorics."
From Reviewer 1 of the *EuroComb Times*

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There is $\mathcal{F}$ with 1,2,3 of size $2^{O(\sqrt{n}\log n)}$.

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Set $s = (1/\sqrt{2} + o(1))\sqrt{n \log n}$ and $t = (\sqrt{2} + o(1))\sqrt{n / \log n}$:

$$|\mathcal{F}| \leq 2^{(1/\sqrt{2}+o(1))\sqrt{n \log n} + (\sqrt{2}+o(1))\sqrt{n/\log n} \cdot \log \sqrt{n}} = 2^{(1+o(1))\sqrt{2n \log n}}.$$

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$B \subset [n] \setminus A$, $|B| = k$, $A \cap B = \emptyset$, use 3 to get size $k+1$ set. □

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Apply with $\mathcal{H} =$ k-element sets of $\mathcal{F}$ with $k = \sqrt{n}$ and $\alpha = \binom{k}{2}$.

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What happens if we change 3 so that always the smaller set gets enlarged? (Cf. matroid independent sets.)

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Bounds are quite far!

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$$s_F = (1, 1, 2^1, 2^2, 2^3, \ldots, 2^{k-3}, n/k, n/k).$$

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$\forall F \in \mathcal{F}, h \geq 0$: $|F \cap X_i| \geq n - h - 1$ for at most 2^h of the X_i's.

$$s_F = (1, 1, 2^1, 2^2, 2^3, \ldots, 2^{k-3}, n/k, n/k).$$

$$\mathrm{rk}(\mathcal{F}) = \max \sum_{i=1}^{k} s_F(i) = n/k + \sum_{i=1}^{k-1} 2^{k-i-1} + 1 = (2 + o(1))n/\log n.$$

$$|\mathcal{F}| \leq \sum_{i=0}^{\mathrm{rk}(\mathcal{F})} \binom{n}{i} \leq \left(\frac{en}{2n/k}\right)^{2n/k} \leq 2^{(2+o(1))n \log\log n/\log n}.$$

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Does 2 ever make a difference?

Thank you for your attention!