Coloring Fat Hypergraphs

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ISU Discrete Mathematics Zoominar 2020

Let's color some hypergraphs!

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Before abstract definitions, geometric examples.

1-dimensional problems

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Theorem (1-dim polychromatic coloring)

Every $X \subset \mathbb{R}$ *can be colored with* k *colors such that every interval with* $\geq k$ *points from* X *is polychromatic (i.e., contains all* k *colors).*

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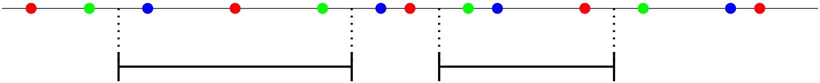

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If collection $\mathcal{I}$ of intervals covers some $X \subset \mathbb{R}$ k-fold, then $\exists$ $\mathcal{I}_1 \cup^* \ldots \cup^* \mathcal{I}_k = \mathcal{I}$ such that each $\mathcal{I}_i$ covers X.

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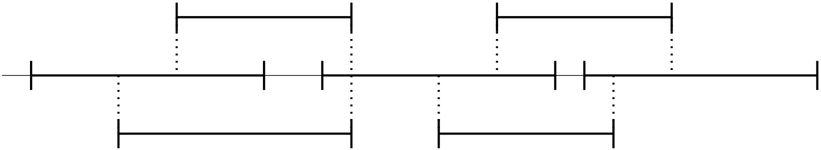

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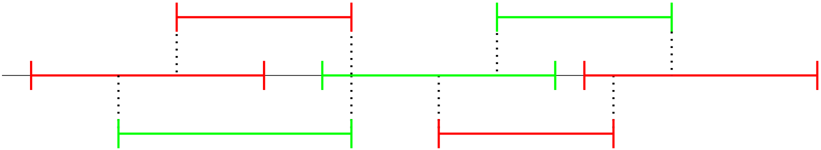

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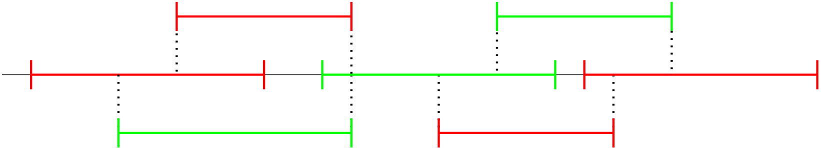

But you picked coloring vertices, so we skip these!

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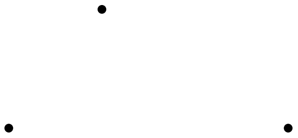

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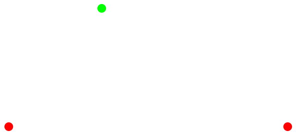

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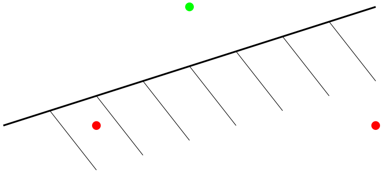

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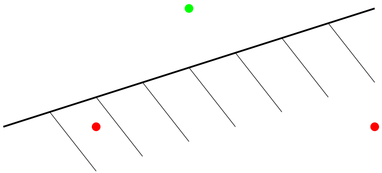

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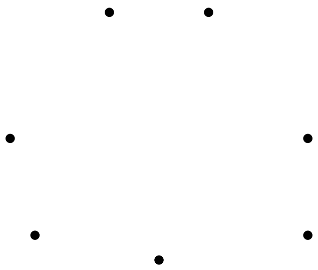

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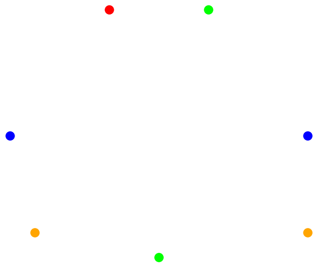

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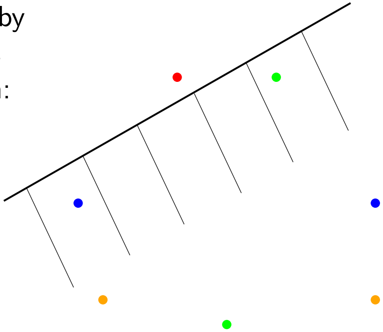

Lines

Theorem (Pach-Tardos-Tóth '09)

For every k, m there is a finite $X \subset \mathbb{R}^2$ such that for every k-coloring of X there is a line that contains at least m points from X and all have the same color.

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From the point-line duality of $\mathbb{R}^2$ we get:

Theorem (Pach-Tardos-Tóth '09)

For every k, m there is a finite collection of lines $\mathcal{L}$ in the plane such that for every k-coloring of $\mathcal{L}$ there is a point contained in at least m lines from $\mathcal{L}$ and all have the same color.

Hereditary k-colorability of abstract hypergraphs

For a hypergraph $\mathcal{H} = (V, \mathcal{E})$, denote by m_k the smallest number for which we can k-color any finite $X \subset V$ such that for any $E \in \mathcal{E}$ with $|E \cap X| \geq m_k$ all k colors occur in $E \cap X$.

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From defs: $\chi_{fat} = 2 \iff m_2 < \infty$

$m_k \leq m_{k+1}$

Berge

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Theorem (Berge '72)

For hereditary families $m_2 = 2$ *if and only if* $m_k = k$ *for every* k*.*

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Proof sketch of $m_2 = 2 \Rightarrow m_3 = 3$.

If some E has two reds but no blues, re-2-color $X =$reds $\cup$ blues.

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What about larger m_2?

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In general, this method gives $m_k \leq (k-1)(m_2-1)+1$.

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What if E' had one red and one blue?

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If some E has two reds but no blues, re-2-color X =reds $\cup$ blues.

What about larger m_2?

In general, this method gives $m_k \leq$ ~~$(k-1)(m_2-1)+1$~~ ~~k^{m_2}~~ ???

Question

Does $m_2 = 3$ imply any upper bound on m_3?

The best constructions we know give $m_3 = 5$:

Halfplanes; Union of two intervals; etc.

Translates in $\mathbb{R}^2$

Conjecture (Pach '80)

For every planar convex set D there is an m such that we can color any finite $X \subset \mathbb{R}^2$ with two colors such that every translate of D with at least m points contains both colors, i.e., $\chi_{fat} = 2$.

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For every convex polygon P and k there is an m_k such that we can color any finite $X \subset \mathbb{R}^2$ with k colors such that every translate of P with at least m_k points contains all k colors, i.e., $m_k < \infty$.

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Above holds with $m_k = O(k)$.

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Theorem (P. '13, Pach-P. '16)

For every m there is a finite $X \subset \mathbb{R}^2$ such that for every two-coloring of X there is a unit disk that contains at least m points from X and all have the same color, i.e., $\chi_{fat} > 2$.

Proper four-coloring

Observation (Cardinal-Korman '13)

For any planar convex set D, any finite $X \subset \mathbb{R}^2$ can be 4-colored such that any translate of D with at least 2 points is non-monochromatic, therefore, $\chi_{fat} \leq 4$.

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Generalized Delaunay triangulation: Connect two points of X if there is a homothet of D that contains only them from X.

Homothet of D: Translated, scaled copy of D. (No rotations!)

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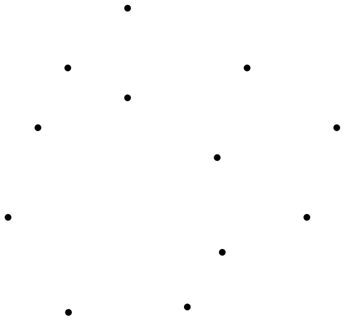

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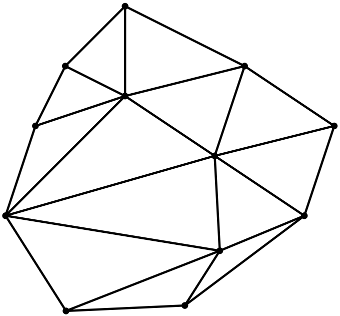

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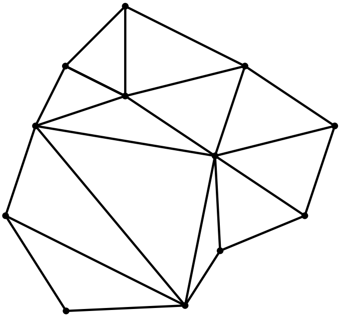

Generalized Delaunay triangulation if D is square.

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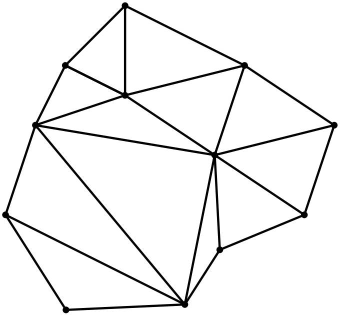

Lemma: Generalized Delaunay triangulation is planar graph.

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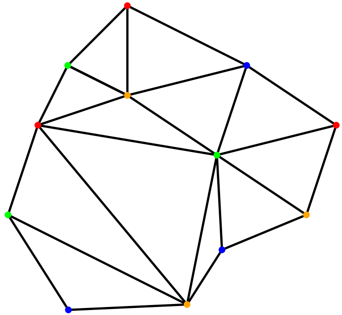

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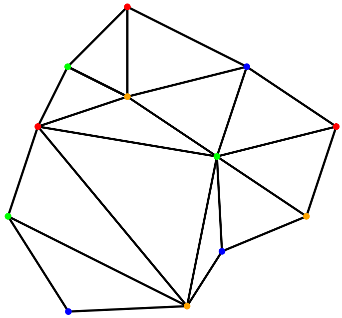

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There is an m such that any finite $X \subset \mathbb{R}^2$ can be 3-colored such that every disk with m points is non-monochromatic, i.e., $\chi_{fat} \leq 3$.

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Could even be true with 2 colors for homothets of convex polygons; triangles (Keszegh-P. '11), squares (Ackerman-Keszegh-Vizer '16). For convex polygons $\chi_{fat} = 2 \Rightarrow m_k = poly(k)$ (Keszegh-P. '14).

Constructions for polygons

Theorem (Pach-Tardos-Tóth '05)

For every non-convex quadrilateral Q and number m there is a finite $X \subset \mathbb{R}^2$ such that for every two-coloring of X there is a translate of Q that contains at least m points from X and all have the same color, i.e., $\chi_{fat} > 2$.

Constructions for polygons

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For translates of convex polygons $m_k = O(k)$.

Question

What about homothets other than triangle and square?

Summary for polygons

	translates	**homothets**
triangles	$\chi_{fat} = 2$ Tardos-Tóth '07 $m_k = O(k)$ Gibson-Varadarajan '11	$\chi_{fat} = 2$ $m_k = O(k^{4.09})$ Cardinal, Knauer, Micek, Ueckerdt '13 + K.-P. '15
convex polygons	$\chi_{fat} = 2$ $m_k = O(k)$ Gibson-Varadarajan '11	$2 \leq \chi_{fat} \leq 3$ Keszegh-P. '17
non-convex polygons*	$3 \leq \chi_{fat} < \infty$ P. '10 + Kovács '17	$3 \leq \chi_{fat} < \infty$ Kovács '17

Constructions for disks

Theorem (Pach-Tardos-Tóth '05)

For every m there is a finite $X \subset \mathbb{R}^2$ *such that for every two-coloring of* X *there is a disk that contains at least m points from* X *and all have the same color, i.e.,* $\chi_{fat} > 2$.

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For every m there is a finite $X \subset \mathbb{R}^2$ *such that for every THREE-coloring of X there is a disk that contains at least m points from X and all have the same color, i.e.,* $\chi_{fat} > 3$.

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So from Delaunay-triangulation we know $\chi_{fat} = 4$.

Stabbed disks

Theorem (Ackerman-Keszegh-P. '19)

For every m there is a finite $X \subset \mathbb{R}^2$ such that in every two-coloring of X there is a pseudo-disk containing the origin and m points from X that is monochromatic, i.e., $\chi_{fat} > 2$.

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Summary for disks

	unit disks	any disks
stabbed	$\chi_{fat} = 2$ $m_k = O(k)$ Damásdi-P. '20	$\chi_{fat} = 3$ Damásdi-P. '20 Ackerman-Keszegh-P. '19
all	$3 \leq \chi_{fat} \leq 4$ P. '13, Pach-P. '16	$\chi_{fat} = 4$ Damásdi-P. '20

Main new proof

Theorem (Damásdi-P. '20)

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Definition

T_m is an m-deep m-ary tree.
$\mathcal{H}_m$ is m-uniform: syblings and root-to-leaf paths in T_m.

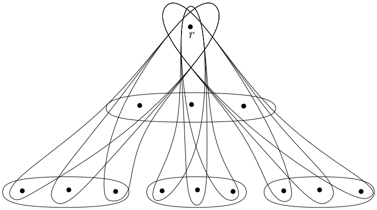

Figure: $\mathcal{H}_3$

Realizing $\mathcal{H}_3$

Initially, put all vertices on a circle, then move them a bit.

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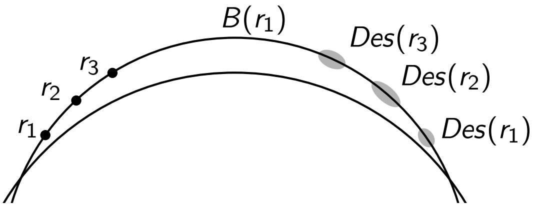

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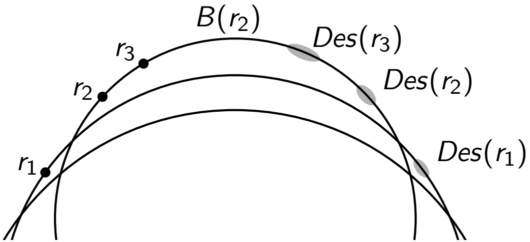

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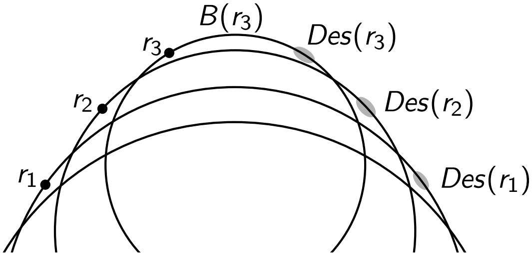

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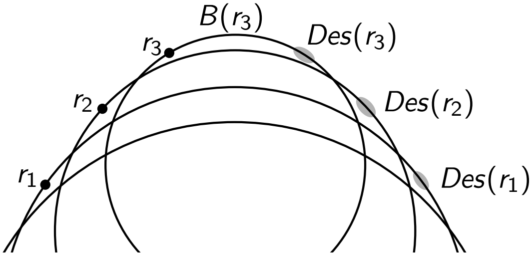

This proves $\chi_{fat} = 3$ for stabbed disks.

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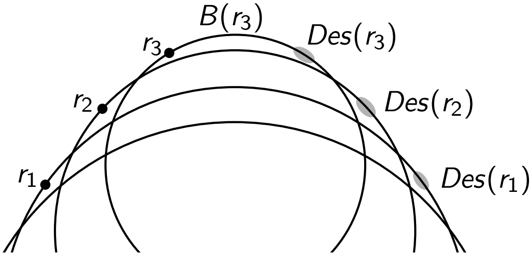

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Then combine with Pach-Tardos-Tóth '05 to get $\chi_{fat} = 4$. □

Summary for disks

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Three dimensions

Theorem (P. '10)

For every polyhedron $P \subset \mathbb{R}^3$ *and* m *there is a finite* $X \subset R^3$ *such that for every two-coloring of* X *there is a monochromatic translate of* P *that contains at least* m *points from* X*, i.e.,* $\chi_{fat} > 2$.

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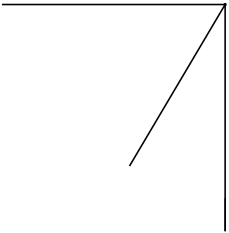
Theorem (Keszegh-P. '11, '15)

Any finite $X \subset \mathbb{R}^3$ can be two-colored such that any translate of an octant with 9 points is non-monochromatic, therefore, $\chi_{fat} = 2$.



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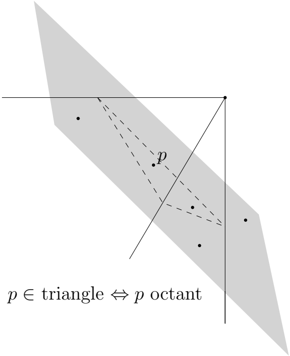
Proof: Embed plane into $\mathbb{R}^3$ as $x + y + z = 0$.



$p \in$ triangle $\Leftrightarrow$ p octant

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Corollary

Any finite $X \subset \mathbb{R}^2$ can be two-colored such that any axis-parallel bottomless rectangle with 9 points is non-monochromatic.

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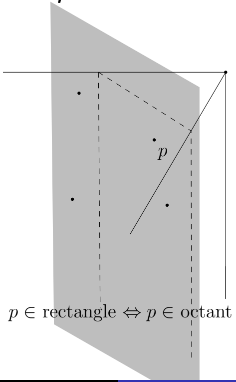

Polychromatic k-colorings

Theorem (Cardinal, Knauer, Micek, Ueckerdt (+ KP) '15)

Any finite $X \subset \mathbb{R}^3$ can be k-colored such that any translate of an octant with $m_k = O(k^{5.09})$ points contains all k colors.

Corollary

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Theorem (Asinowski et al. '13)

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Any finite $X \subset \mathbb{R}^3$ can be k-colored such that any translate of an octant with $m_k = O(k^{5.09})$ points contains all k colors.

Corollary

Any finite $X \subset \mathbb{R}^2$ can be k-colored such that any homothet of a triangle with $m_k = O(k^{5.09})$ points contains all k colors.

Theorem (Asinowski et al. '13)

Any finite $X \subset \mathbb{R}^2$ can be k-colored such that any axis-parallel bottomless rectangle with $m_k \leq 3k - 2$ points contains all k colors.

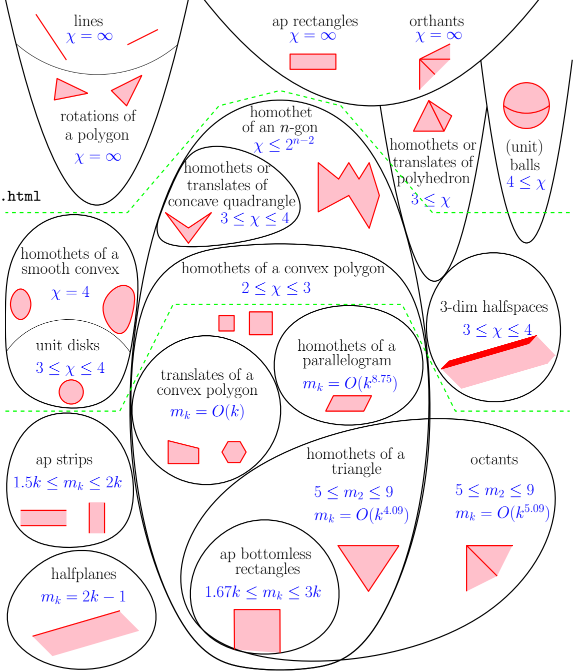
None of the above is known to be sharp.

Summary

	translates	homothets
triangles	$\chi_{fat} = 2$ $m_k = O(k)$	$\chi_{fat} = 2$ $m_k = O(k^{4.09})$
convex polygons	$\chi_{fat} = 2$ $m_k = O(k)$	$2 \leq \chi_{fat} \leq 3$
non-convex polygons*	$3 \leq \chi_{fat} < \infty$	$3 \leq \chi_{fat} < \infty$
stabbed convex polygons	$\chi_{fat} = 2$ $m_k = O(k)$	$\chi_{fat} = 2$ $m_k = O(k)$
stabbed disks	$\chi_{fat} = 2$ $m_k = O(k)$	$\chi_{fat} = 3$
disks	$3 \leq \chi_{fat} \leq 4$	$\chi_{fat} = 4$

Summary of known results

for all results, see
`coge.elte.hu/cogezoo.html`



Thank you for your attention!