Weak embeddings of posets to the Boolean lattice

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From Sperner's theorem

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If $\mathcal{F} \subset 2^{[n]}$ *is* $(k+1)$*-chain-free, then* $|\mathcal{F}| \le k\binom{n}{\lfloor n/2 \rfloor}$*, extremal* $\mathcal{F}$ *consists of* k *middle levels (slightly less).*

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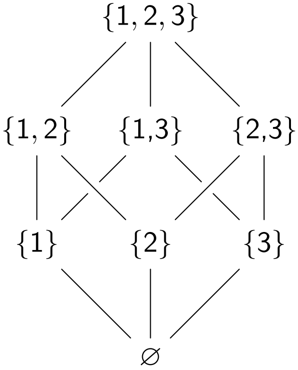
Problem (Katona '80s)

What if we forbid other configurations?

Boolean lattice and other posets

The collection of all subsets of an n-element set is the poset B_n with the containment relation $\subseteq$.
Every collection of sets $P \subseteq 2^{[n]}$ is also a poset with relation $\subseteq$.
Below is B_3:



Poset embedding

Given two posets, P and Q, we say that P is a *subposet* of Q if there exists an injection (weak embedding) $\phi : P \to Q$ such that $x \leq y$ in P implies $\phi(x) \leq \phi(y)$ in Q.
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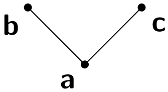
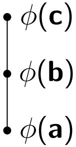
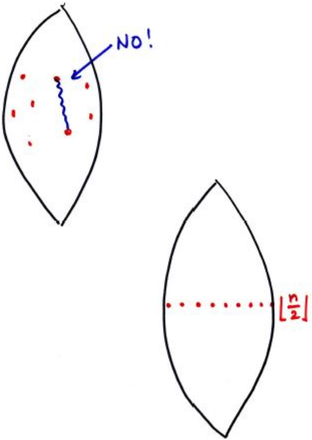


is a subposet of a chain, by the mapping ϕ:



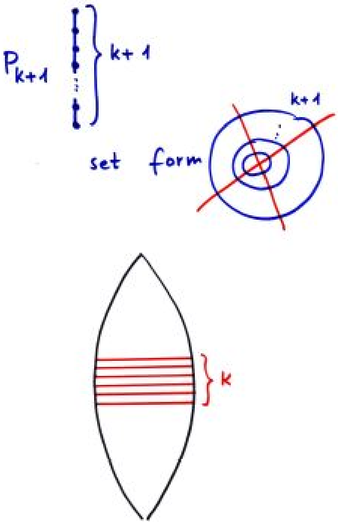
Sperner's theorem

Inclusion-free families in the Boolean lattice $B_n = (2^{[n]}, \subseteq)$.



(Illustrations by Gyula O.H. Katona and many slides from Abhishek Methuku.)

Erdős's theorem



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⋮

Bukh '09: $\mathrm{La}(n, T) \approx (h(T) - 1)\binom{n}{\lfloor \frac{n}{2} \rfloor}$ if the Hasse-diagram of the poset T is a tree, and $h(T)$ denotes its height.

Main Conjecture

Conjecture (Everyone*)

$$\pi(P) := \lim_{n\to\infty} \frac{\mathrm{La}(n,P)}{\binom{n}{\lfloor \frac{n}{2} \rfloor}}$$

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Example. $\pi(\bowtie) = e(\bowtie) = 2 < 3 = \min\{n \mid \bowtie \subset B_n\}$.

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- Also NP-complete whether a graph is an induced subgraph of a Johnson-graph (all vertices on single level, connect two if their distance equals 2).

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Strong embeddings

Given two posets, P and Q, we say that P is an *induced subposet* of Q if there exists an injection (strong embedding) $\phi : P \to Q$ such that $x \leq y$ in P if and only if $\phi(x) \leq \phi(y)$ in Q.

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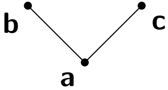
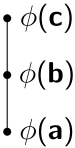


is NOT a subposet of a chain.



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Most of our hardness results remain valid for strong embeddings.

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Conjecture (P. et possibly al.): $n \approx 1.29d$.

Thank you for your attention!