Adaptive Majority Problems for Graphs

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Joint work with
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balls	queries
1	0
2	1
3	1
4	3
5	3
6	4
7	4
8	7
9	7
10	8
11	8
12	10

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1	0	1	0
2	1	10	0
3	1	11	1
4	3	100	0
5	3	101	1
6	4	110	1
7	4	111	2
8	7	1000	0
9	7	1001	1
10	8	1010	1
11	8	1011	2
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Theorem (Saks and Werman '91)

Best algorithm finds majority ball with $n - b(n)$ *queries.*

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TOPIC OF TALK: OTHER CONNECTED GRAPHS

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Observation

$n - 1 - \log_2 n \leq n - b(n) = m(K_n) \leq m(G) \leq n - 1.$

Results

Theorem (DGKKLMNPPVW)

For every tree T on an even number n of vertices $m(T) = n - 1$
and
for every tree T on an odd number n of vertices $m(T) \geq n - 65$.

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Question

Superlinear lower bound or $O(n)$ edge construction?

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Theorem (DGKKLMNPPVW)

For odd paths, non-deterministic query complexity is $n - \Theta(\sqrt{n})$.

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Lemma

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(i) *If* $w_1 = .. = w_{2^t} = 1$ *and* $\sum_{i=1}^{k} w_i = 2^{t+1}$*, then* $m(\underline{w}) = k - 1$.
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Proof easy using Rivest–Vuillemin theorem about decision trees.

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Corollary

Let $w_1, \dots, w_j$ be each powers of two and $k > 2^t$ $(+1$ if k is odd$)$.

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Proof.

Imagine that we start queries for Lemma like this. □

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True for $n < 1000$ as $m(P_n) \geq n - b(n)$. Let U include every 9^{th} vertex of P_n, so $|U| = \frac{n}{9}$, and $P_n \setminus U$ consists of paths on 8 vertices.

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if $X \cap U = \emptyset$, then $w(X) \leq 1$,
if $X \cap U \neq \emptyset$, then $1 \leq w(X) \leq 2|X \cap U|$.

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$\sum_{X:X\cap U\neq\emptyset} w(X) \leq 2|U| = 2\frac{n}{9} < 2^t$, so $\sum_{X:X\cap U=\emptyset} 1 \geq 2^t$.

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Proposition

There is U of at most $n/16$ vertices such that every component of $T \setminus U$ has at most 32 edges (including edges connecting to U).

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(i) *if $|X|$ is odd, $w(X) = 1$, and*
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where $\delta(X) =$ parity of the number of components of $V \setminus X$.

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Thank you for your attention!