Monochromatic configurations on a circle

Dömötör Pálvölgyi

ELTE Eötvös Loránd University, Budapest

Joint work with
Gábor Damásdi and Nóra Frankl and János Pach

EuroComb 2023, Prague

Motivation

Problem 12251 (Robert Tauraso, Amer. Math. Monthly '21)

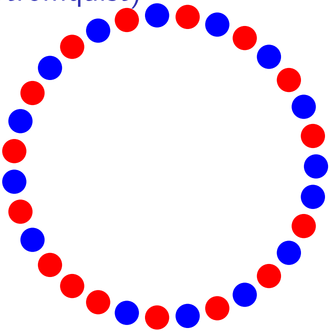
If all the points of the plane are colored blue or red, find a unit area convex pentagon with all vertices of the same color.

Motivation

Problem 12251 (Robert Tauraso, Amer. Math. Monthly '21)

If all the points of the plane are colored blue or red, find a unit area convex pentagon with all vertices of the same color.

Solution (Walter Stromquist)



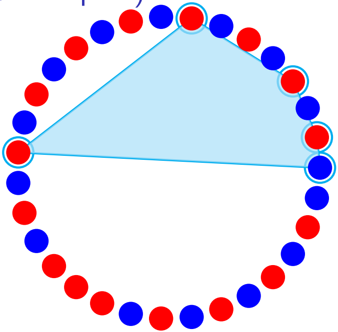
Take regular 31-gon, with all vertices colored blue or red.

Motivation

Problem 12251 (Robert Tauraso, Amer. Math. Monthly '21)

If all the points of the plane are colored blue or red, find a unit area convex pentagon with all vertices of the same color.

Solution (Walter Stromquist)



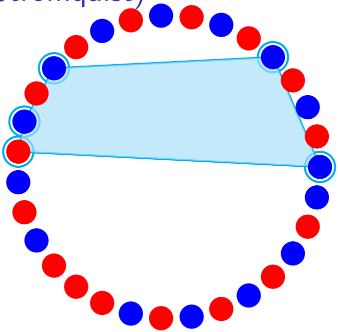
Take 5 points that divide the circle as $1 : 2 : 4 : 8 : 16$.

Motivation

Problem 12251 (Robert Tauraso, Amer. Math. Monthly '21)

If all the points of the plane are colored blue or red, find a unit area convex pentagon with all vertices of the same color.

Solution (Walter Stromquist)



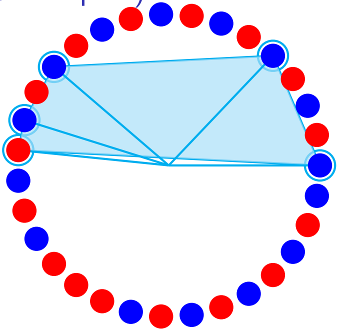
The arcs need not be in order 1 : 2 : 4 : 8 : 16

Motivation

Problem 12251 (Robert Tauraso, Amer. Math. Monthly '21)

If all the points of the plane are colored blue or red, find a unit area convex pentagon with all vertices of the same color.

Solution (Walter Stromquist)



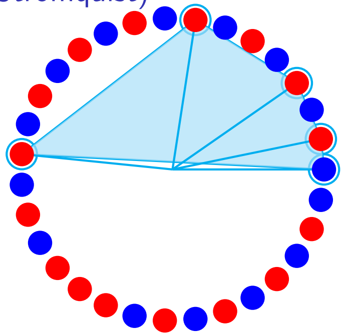
Area of pentagon is always the same.

Motivation

Problem 12251 (Robert Tauraso, Amer. Math. Monthly '21)

If all the points of the plane are colored blue or red, find a unit area convex pentagon with all vertices of the same color.

Solution (Walter Stromquist)



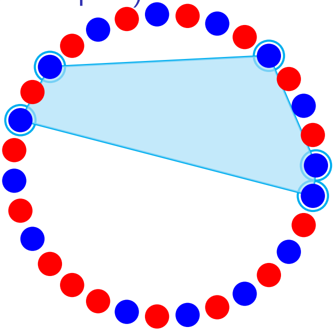
Area of pentagon is always the same.

Motivation

Problem 12251 (Robert Tauraso, Amer. Math. Monthly '21)

If all the points of the plane are colored blue or red, find a unit area convex pentagon with all vertices of the same color.

Solution (Walter Stromquist)



Thus, we only need 5 monochromatic points as $1:2:4:8:16$.

Motivation

Problem 12251 (Robert Tauraso, Amer. Math. Monthly '21)

If all the points of the plane are colored blue or red, find a unit area acute pentagon with all vertices of the same color.

Solution (Walter Stromquist)

We can always find 5 points with the same color that divide any fixed two-colored circle into arcs proportional to $1 : 2 : 4 : 8 : 16$.

Motivation

Problem 12251 (Robert Tauraso, Amer. Math. Monthly '21)

If all the points of the plane are colored blue or red, find a unit area acute pentagon with all vertices of the same color.

Solution (Walter Stromquist)

We can always find 5 points with the same color that divide any fixed two-colored circle into arcs proportional to $1 : 2 : 4 : 8 : 16$.

Proof (Stromquist).

Check all cases with a computer. □

Motivation

Problem 12251 (Robert Tauraso, Amer. Math. Monthly '21)

If all the points of the plane are colored blue or red, find a unit area acute pentagon with all vertices of the same color.

Solution (Walter Stromquist)

We can always find 5 points with the same color that divide any fixed two-colored circle into arcs proportional to $1 : 2 : 4 : 8 : 16$.

Proof (Stromquist).

Check all cases with a computer. □

Conjecture (Stromquist, for all k)

We can always find k points with the same color that divide any fixed two-colored circle into arcs proportional to $1 : 2 : 4 : .. : 2^{k-1}$.

Why powers of 2?

Conjecture (Stromquist, for all k)

We can always find k points with the same color that divide any fixed two-colored circle into arcs proportional to $1 : 2 : 4 : .. : 2^{k-1}$.

Why powers of 2?

Conjecture (Stromquist, for all k)

We can always find k points with the same color that divide any fixed two-colored circle into arcs proportional to $1:2:4:..:2^{k-1}$.

Theorem (Damásdi, N. Frankl, Pach, P.)

For ratios not equivalent to $1:2:4:..:2^{k-1}$, *there is a 2-coloring of the circle with no monochromatic k-gon with given proportions.*

Why powers of 2?

Conjecture (Stromquist, for all k)

We can always find k points with the same color that divide any fixed two-colored circle into arcs proportional to $1 : 2 : 4 : .. : 2^{k-1}$.

Theorem (Damásdi, N. Frankl, Pach, P.)

For ratios not equivalent to $1 : 2 : 4 : .. : 2^{k-1}$, there is a 2-coloring of the circle with no monochromatic k-gon with given proportions.

In fact, we can even take a uniform 2-coloring.

Why powers of 2?

Conjecture (Stromquist, for all k)

We can always find k points with the same color that divide any fixed two-colored circle into arcs proportional to $1:2:4:..:2^{k-1}$.

Theorem (Damásdi, N. Frankl, Pach, P.)

For ratios not equivalent to $1:2:4:..:2^{k-1}$, *there is a 2-coloring of the circle with no monochromatic k-gon with given proportions.*

In fact, we can even take a uniform 2-coloring.

Conjecture is still open; we verified it for $k \leq 7$.

Beatty sequences

A Beatty sequence is a sequence $\{\lfloor \alpha n + \beta \rfloor\}_{n=1}^{\infty}$ where $\alpha, \beta \in \mathbb{R}$.

Beatty sequences

A Beatty sequence is a sequence $\{\lfloor \alpha n + \beta \rfloor\}_{n=1}^{\infty}$ where $\alpha, \beta \in \mathbb{R}$.
Denote $B(\alpha, \beta) = \{\lfloor \alpha n + \beta \rfloor\}_{n=1}^{\infty} \subset \mathbb{N}$.

Beatty sequences

A Beatty sequence is a sequence $\{\lfloor \alpha n + \beta \rfloor\}_{n=1}^{\infty}$ where $\alpha, \beta \in \mathbb{R}$.
Denote $B(\alpha, \beta) = \{\lfloor \alpha n + \beta \rfloor\}_{n=1}^{\infty} \subset \mathbb{N}$.

Question

Which $B(\alpha_1, \beta_1), \ldots, B(\alpha_k, \beta_k)$ partition $\mathbb{N}$?

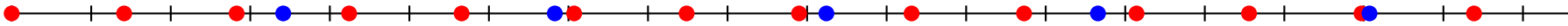
Beatty sequences

A Beatty sequence is a sequence $\{\lfloor \alpha n + \beta \rfloor\}_{n=1}^{\infty}$ where $\alpha, \beta \in \mathbb{R}$.
Denote $B(\alpha, \beta) = \{\lfloor \alpha n + \beta \rfloor\}_{n=1}^{\infty} \subset \mathbb{N}$.

Question

Which $B(\alpha_1, \beta_1), \ldots, B(\alpha_k, \beta_k)$ partition $\mathbb{N}$?

Folklore: If α_1,α_2 are positive irrationals such that $\frac{1}{\alpha_1} + \frac{1}{\alpha_2} = 1$, then $B(\alpha_1, 0) \,\dot\cup\, B(\alpha_2, \alpha_2) = \mathbb{N}$.



Beatty sequences

A Beatty sequence is a sequence $\{\lfloor \alpha n + \beta \rfloor\}_{n=1}^{\infty}$ where $\alpha, \beta \in \mathbb{R}$.
Denote $B(\alpha, \beta) = \{\lfloor \alpha n + \beta \rfloor\}_{n=1}^{\infty} \subset \mathbb{N}$.

Question

Which $B(\alpha_1, \beta_1), \dots, B(\alpha_k, \beta_k)$ partition $\mathbb{N}$?

Folklore: If α_1,α_2 are positive irrationals such that $\frac{1}{\alpha_1} + \frac{1}{\alpha_2} = 1$, then $B(\alpha_1, 0) \,\dot\cup\, B(\alpha_2, \alpha_2) = \mathbb{N}$.

Conjecture (Fraenkel's conjecture)

If $B(\alpha_1, \beta_1) \,\dot\cup\, \dots \,\dot\cup\, B(\alpha_k, \beta_k) = \mathbb{N}$ for $0 < \alpha_1 < \dots < \alpha_k$ and $k \geq 3$, then $\alpha_i = \frac{2^k - 1}{2^{i-1}}$ for $1 \leq i \leq k$.

Beatty sequences

A Beatty sequence is a sequence $\{\lfloor \alpha n + \beta \rfloor\}_{n=1}^{\infty}$ where $\alpha, \beta \in \mathbb{R}$.
Denote $B(\alpha, \beta) = \{\lfloor \alpha n + \beta \rfloor\}_{n=1}^{\infty} \subset \mathbb{N}$.

Question

Which $B(\alpha_1, \beta_1), \dots, B(\alpha_k, \beta_k)$ partition $\mathbb{N}$?

Folklore: If α_1,α_2 are positive irrationals such that $\frac{1}{\alpha_1} + \frac{1}{\alpha_2} = 1$, then $B(\alpha_1, 0) \dot\cup B(\alpha_2, \alpha_2) = \mathbb{N}$.

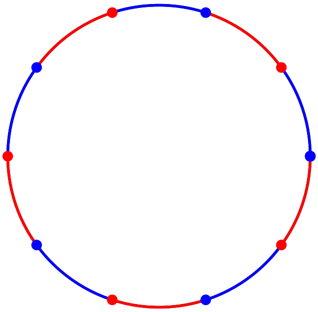
Conjecture (Fraenkel's conjecture)

If $B(\alpha_1, \beta_1) \dot\cup \dots \dot\cup B(\alpha_k, \beta_k) = \mathbb{N}$ for $0 < \alpha_1 < \dots < \alpha_k$ and $k \geq 3$, then $\alpha_i = \frac{2^k - 1}{2^{i-1}}$ for $1 \leq i \leq k$.

Known to hold (coincidentally also) for $k \leq 7$ (Barát, Varjú '03).

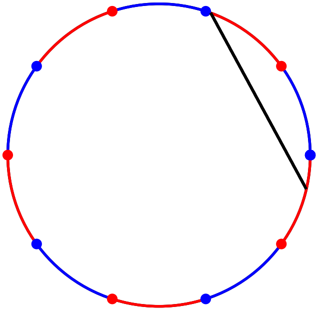
Connection

Connection



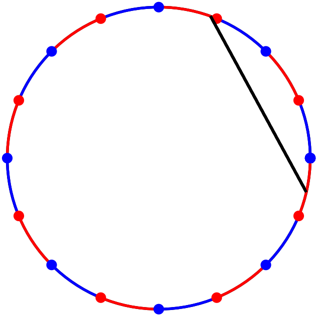
Take a uniform 2-coloring of the circle that consists of $2t$ arcs of length $\frac{1}{2t}$, colored blue and red in an alternating manner.

Connection



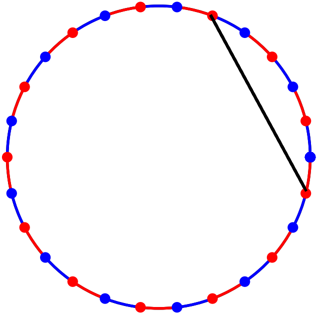
An arc of length d with red endpoints jumps over $\lfloor td \rceil$ blue intervals, where $\lfloor x \rceil$ is the rounding of x to the nearest integer.

Connection



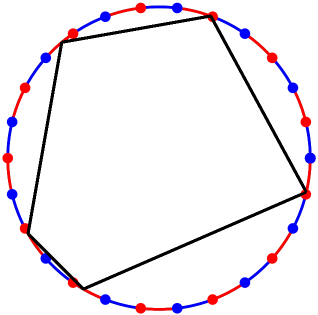
An arc of length d with red endpoints jumps over $\lfloor td \rceil$ blue intervals, where $\lfloor x \rceil$ is the rounding of x to the nearest integer.

Connection



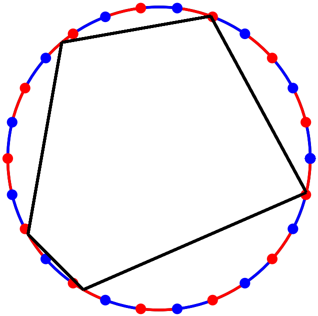
An arc of length d with red endpoints jumps over $\lfloor td \rceil$ blue intervals, where $\lfloor x \rceil$ is the rounding of x to the nearest integer.

Connection



All the arcs together need to jump over exactly all t blue intervals.
This implies $\sum_{i=1}^{k} \lfloor t d_i \rceil = t$ for every $t \in \mathbb{N}$.

Connection



All the arcs together need to jump over exactly all t blue intervals.
This implies $\sum_{i=1}^{k} \lfloor td_i \rceil = t$ for every $t \in \mathbb{N}$.

$$\sum_{i=1}^{k} \left(\lfloor td_i \rceil - \lfloor (t-1)d_i \rceil \right) = t - (t-1) = 1.$$

Connection

All the arcs together need to jump over exactly all t blue intervals. This implies $\sum_{i=1}^{k} \lfloor td_i \rceil = t$ for every $t \in \mathbb{N}$.

$$\sum_{i=1}^{k} (\lfloor td_i \rceil - \lfloor (t-1)d_i \rceil) = t - (t-1) = 1.$$

In other words, exactly one of $\lfloor td_i \rceil - \lfloor (t-1)d_i \rceil = 1$.

Connection

All the arcs together need to jump over exactly all t blue intervals. This implies $\sum_{i=1}^{k} \lfloor td_i \rceil = t$ for every $t \in \mathbb{N}$.

$$\sum_{i=1}^{k} (\lfloor td_i \rceil - \lfloor (t-1)d_i \rceil) = t - (t-1) = 1.$$

In other words, exactly one of $\lfloor td_i \rceil - \lfloor (t-1)d_i \rceil = 1$.

$\lfloor td_i \rceil - \lfloor (t-1)d_i \rceil = 1$ exactly when t is in the sequence $\{\lfloor (n+\frac{1}{2})\frac{1}{d_i} \rfloor\}_{n=1}^{\infty} = \{\lfloor n\frac{1}{d_i} + \frac{1}{2d_i} \rfloor\}_{n=1}^{\infty} = B(\frac{1}{d_i}, \frac{1}{2d_i})$.

Connection

All the arcs together need to jump over exactly all t blue intervals. This implies $\sum_{i=1}^{k} \lfloor td_i \rceil = t$ for every $t \in \mathbb{N}$.

$$\sum_{i=1}^{k} (\lfloor td_i \rceil - \lfloor (t-1)d_i \rceil) = t - (t-1) = 1.$$

In other words, exactly one of $\lfloor td_i \rceil - \lfloor (t-1)d_i \rceil = 1$.

$\lfloor td_i \rceil - \lfloor (t-1)d_i \rceil = 1$ exactly when t is in the sequence $\{\lfloor (n+\frac{1}{2})\frac{1}{d_i} \rfloor\}_{n=1}^{\infty} = \{\lfloor n\frac{1}{d_i} + \frac{1}{2d_i} \rfloor\}_{n=1}^{\infty} = B(\frac{1}{d_i}, \frac{1}{2d_i})$.

$$B(\frac{1}{d_1}, \frac{1}{2d_1}) \,\dot\cup\, \ldots \,\dot\cup\, B(\frac{1}{d_k}, \frac{1}{2d_k}) = \mathbb{N}.$$

Special case of Fraenkel's conjecture

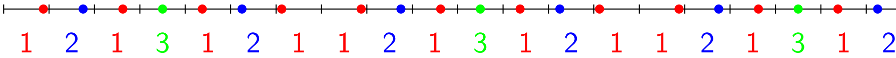
$$B(\frac{1}{d_1}, \frac{1}{2d_1}) \dot\cup \ \dots \ \dot\cup \ B(\frac{1}{d_k}, \frac{1}{2d_k}) = \mathbb{N}.$$

Special case of Fraenkel's conjecture

$$B\left(\frac{1}{d_1}, \frac{1}{2d_1}\right) \,\dot{\cup}\, \ldots \,\dot{\cup}\, B\left(\frac{1}{d_k}, \frac{1}{2d_k}\right) = \mathbb{N}.$$

Special case of Fraenkel's conjecture

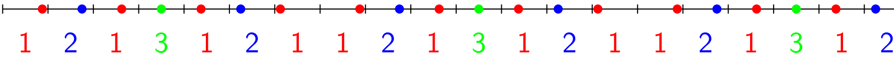
$$B(\frac{1}{d_1}, \frac{1}{2d_1}) \dot\cup \ \dots \ \dot\cup \ B(\frac{1}{d_k}, \frac{1}{2d_k}) = \mathbb{N}.$$



For every $j \in \mathbb{N}$, let $s_j = i$ if $j \in B(\frac{1}{d_i}, \frac{1}{2d_i})$.

Special case of Fraenkel's conjecture

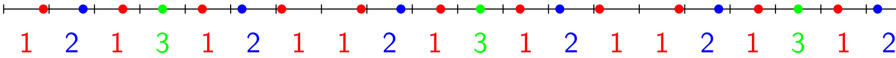
$$B(\frac{1}{d_1}, \frac{1}{2d_1}) \dot{\cup} \ \dots \ \dot{\cup} \ B(\frac{1}{d_k}, \frac{1}{2d_k}) = \mathbb{N}.$$



For every $j \in \mathbb{N}$, let $s_j = i$ if $j \in B(\frac{1}{d_i}, \frac{1}{2d_i})$.
This sequence is balanced with densities $d_1, \dots, d_k$.
Balanced: in any two subsequences of same length the number of any given symbol differs by at most one.

Special case of Fraenkel's conjecture

$$B(\frac{1}{d_1}, \frac{1}{2d_1}) \dot\cup \ \ldots \ \dot\cup \ B(\frac{1}{d_k}, \frac{1}{2d_k}) = \mathbb{N}.$$



For every $j \in \mathbb{N}$, let $s_j = i$ if $j \in B(\frac{1}{d_i}, \frac{1}{2d_i})$.

This sequence is balanced with densities $d_1, \ldots, d_k$.

Balanced: in any two subsequences of same length the number of any given symbol differs by at most one.

Lemma (Altman, Gaujal, Hordijk 2000)

If for every $1 \leq i \leq k$ there exist two consecutive i in S with no j between them for any $j > i$, then densities are $d_i = \frac{2^{i-1}}{2^k - 1}$. □

Steinitz lemma

Conjecture (Stromquist, for all k)

We can always find k points with the same color that divide any fixed two-colored circle into arcs proportional to $1 : 2 : 4 : .. : 2^{k-1}$.

Steinitz lemma

Conjecture (Stromquist, for all k)

We can always find k points with the same color that divide any fixed two-colored circle into arcs proportional to $1 : 2 : 4 : .. : 2^{k-1}$.

Question

Can we always find k such points in uniform coloring?

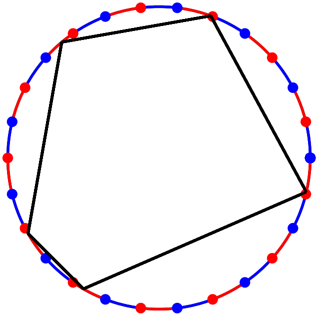
Steinitz lemma

Conjecture (Stromquist, for all k)

We can always find k points with the same color that divide any fixed two-colored circle into arcs proportional to $1 : 2 : 4 : .. : 2^{k-1}$.

Question

Can we always find k such points in uniform coloring?



Steinitz lemma

Conjecture (Stromquist, for all k)

We can always find k points with the same color that divide any fixed two-colored circle into arcs proportional to $1 : 2 : 4 : .. : 2^{k-1}$.

Question

Can we always find k such points in uniform coloring?

Conjecture

If a sequence of reals $-1 < x_1, \ldots, x_k < 1$ *satisfies*

$$x_{i+1} = \begin{cases} 2x_i, & \text{if } 2|x_i| < 1 \\ 2x_i - 2, & \text{if } 2x_i > 1 \\ 2x_i + 2, & \text{if } 2x_i < -1 \end{cases}$$

for $i = 1, \ldots, k$, *where* $x_{k+1} = x_1$, *then there is a permutation* π *of* $\{1, \ldots, k\}$ *such that* $0 \leq \sum_{i=1}^{j} x_{\pi(i)} < 1$ *for every* j.

Density

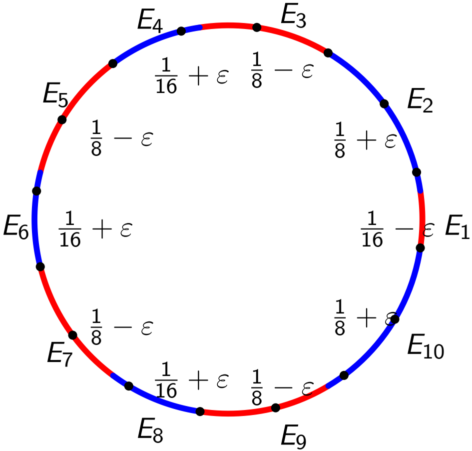
Question

Is it true that the larger color class contains a k-gon?

Density

Question

Is it true that the larger color class contains a k-gon?



No red 6-gon with arcs proportional to 1 : 2 : 4 : 8 : 16 : 32.

Thank you for your attention!

Thank you for your attention!

Conjecture

If a sequence of reals $-1 < x_1, \ldots, x_k < 1$ *satisfies*

$$x_{i+1} = \begin{cases} 2x_i, & \text{if } 2|x_i| < 1 \\ 2x_i - 2, & \text{if } 2x_i > 1 \\ 2x_i + 2, & \text{if } 2x_i < -1 \end{cases}$$

for $i = 1, \ldots, k$, *where* $x_{k+1} = x_1$, *then there is a permutation* π *of* $\{1, \ldots, k\}$ *such that* $0 \leq \sum_{i=1}^{j} x_{\pi(i)} < 1$ *for every* j.