Order types meet Ramsey theory

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Joint work with
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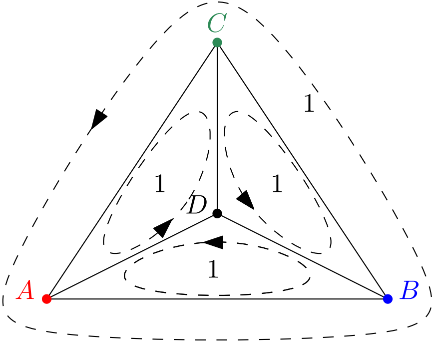

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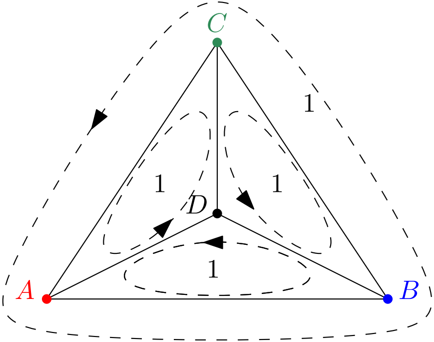

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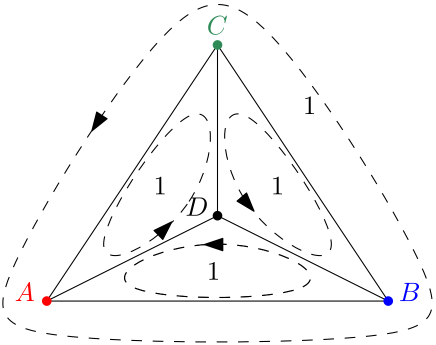

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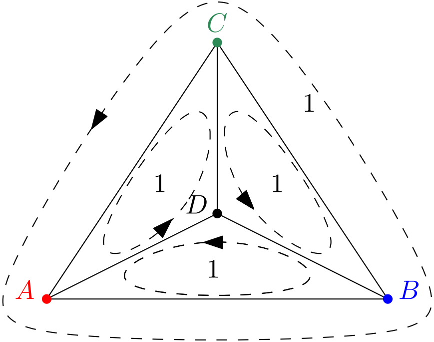

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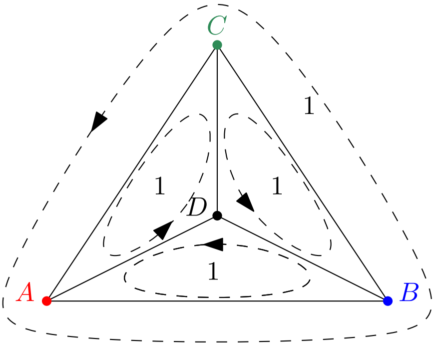

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Much less than $2^{\binom{n}{3}}$.

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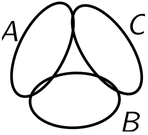

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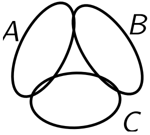

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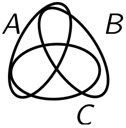

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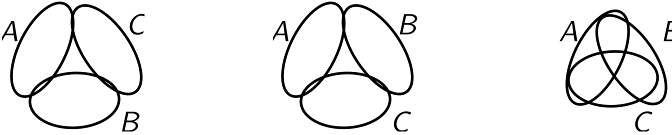

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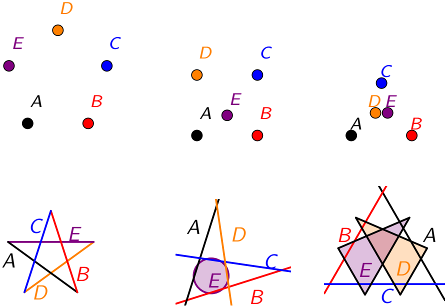

The three non-degenerate five-point order types are realizable with convex sets.

Related work: Orientation of convex sets and good covers

Theorem (Ágoston, Damásdi, Keszegh, P.)

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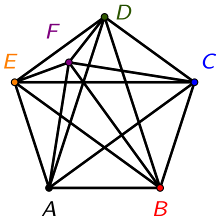
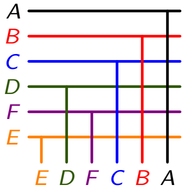


Order type



Good cover realization

Questions about order types

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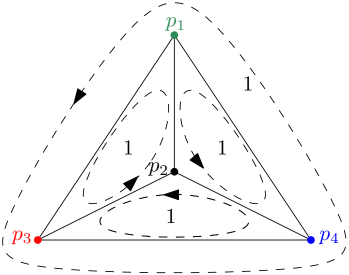

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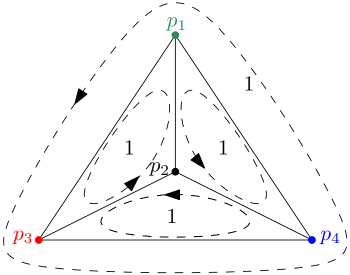

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Simple trick also works for $t > 3$ (another homework).

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So questions for $k = 2$ colors and for more colors are equivalent.

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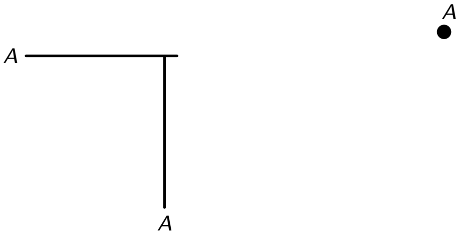

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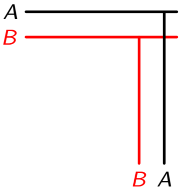

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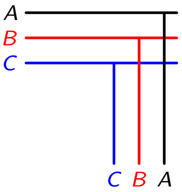

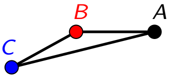

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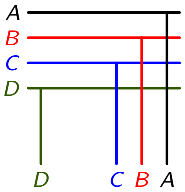

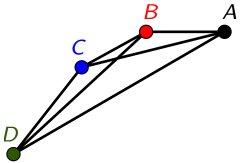

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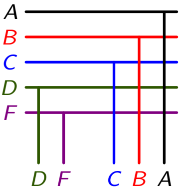

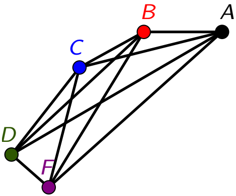

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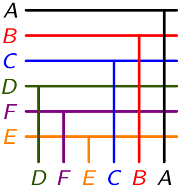

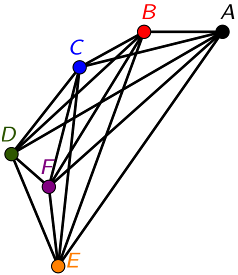

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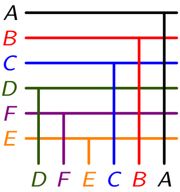

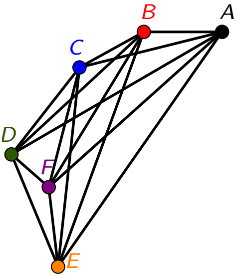

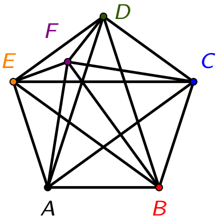

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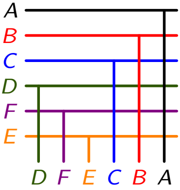

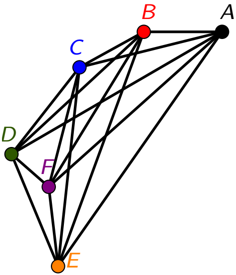

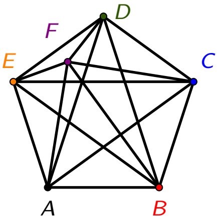

Theorem (Keszegh-P.)

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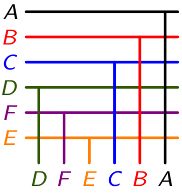

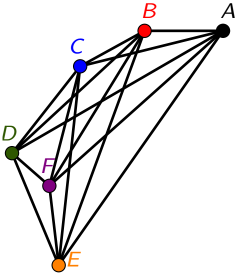

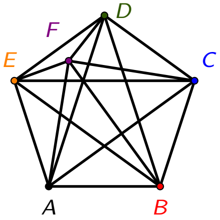

Theorem (Keszegh-P.)

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Corollary

All order types on at most 5 points are Ramsey.

The truth is...

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Not all order types are Ramsey.

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Remark. Statement is simpler if we allow collinearity.

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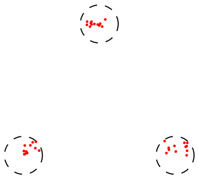

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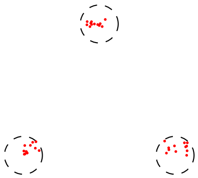

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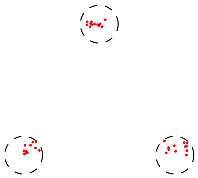

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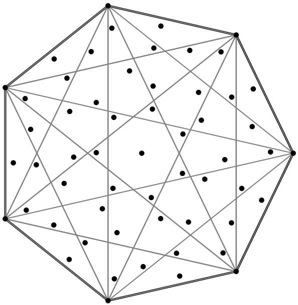
We call these the top level clusters.

Rad realization of n-gon

Consider the order type defined by the vertices of a regular convex n-gon, with one point added to each region defined by its diagonals.



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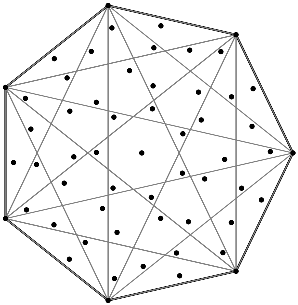

Denote this order type by $\mathcal{Q}_n$.

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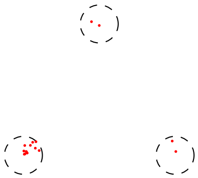
Lemma

In a rad realization of $\mathcal{Q}_n$, there are at most $O(n^{2/3})$ vertices in all but one top level cluster.



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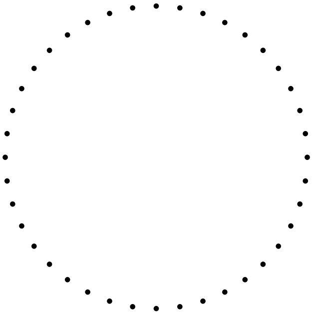

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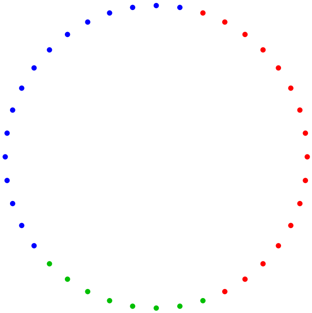

Top level clusters contain consecutive vertices of n-gon.

Rad realization of n-gon

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If two clusters contain $\Omega(n^{2/3})$ vertices, some two clusters contain $\Omega(n^{2/3})$ opposite vertices.

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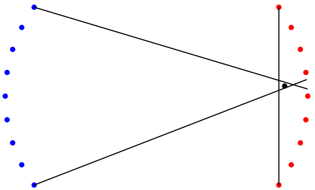

These three diagonals must intersect like that because of extra points in each region and choice of $n^{2/3}$.

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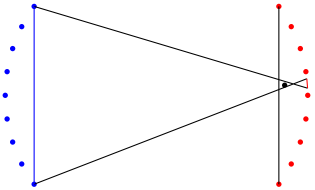

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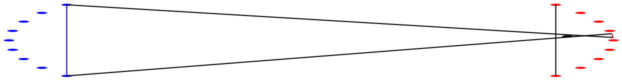

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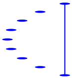

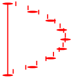

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Repeating this for all the little red segments and summing up, we get that long red segment is shorter than blue segment if 1000 is large enough compared to n.

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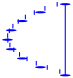

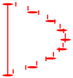

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Swapping the roles of blue and red, we get a contradiction. □

One-by-one rad realization

Similar argument also works if instead of $\mathcal{Q}_n$ from regular n-gon, we take appropriately elongated n-gon.

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Also make $\mathcal{Q}_n$'s between n-gons; main part of each in same cluster.

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This way eventually we get a one-by-one rad realization of $\mathcal{P}$. □

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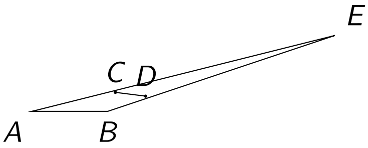
Claim

This has no one-by-one rad realization if in topmost clusters E is alone, then in second clustering D is alone etc., and if ABE is thin.



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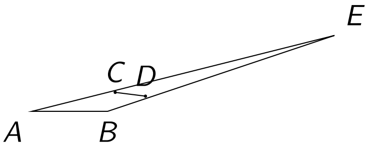

Proof.

CD is shorter than AB but it should be 1000 times longer. □

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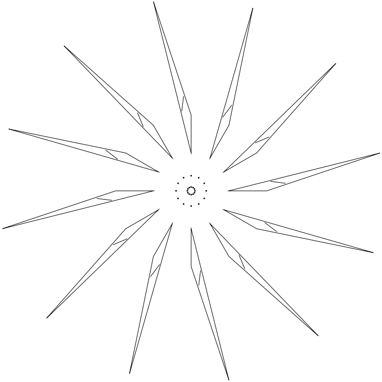

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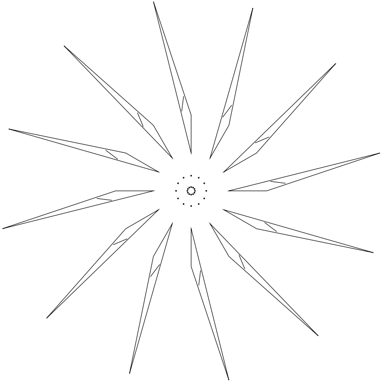

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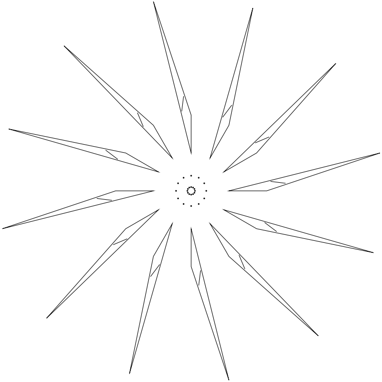

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By pigeonhole principle, one of them is thin. □

Thank you for your attention!