Proper Coloring of Geometric Hypergraphs

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Brisbane

1-dimensional problems

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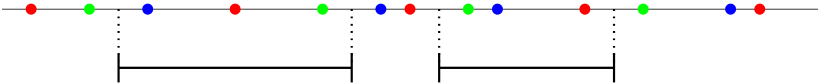

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If collection $\mathcal{I}$ of intervals covers some $X \subset \mathbb{R}$ k-fold, then $\exists\ \mathcal{I}_1 \cup^* \ldots \cup^* \mathcal{I}_k = \mathcal{I}$ such that each $\mathcal{I}_i$ covers X.

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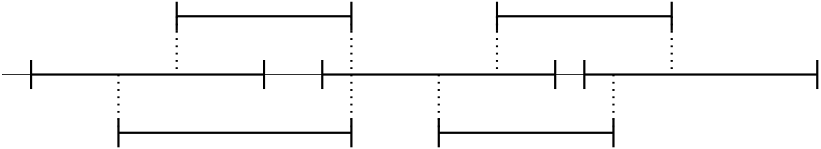

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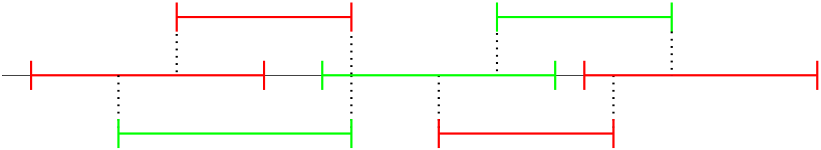

For finite $\mathcal{I}$ greedy algorithm works.

Halfplanes

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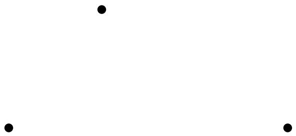

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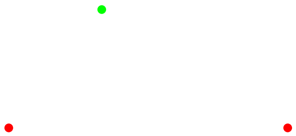

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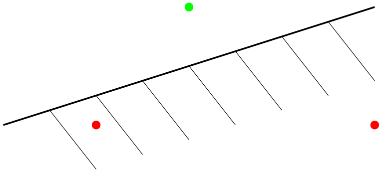

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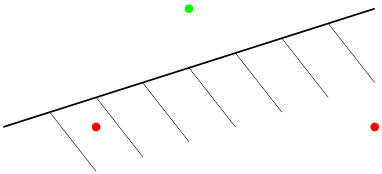

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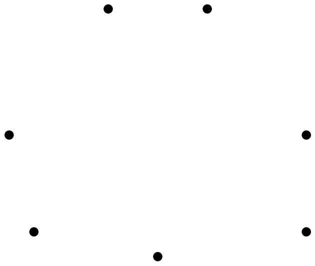

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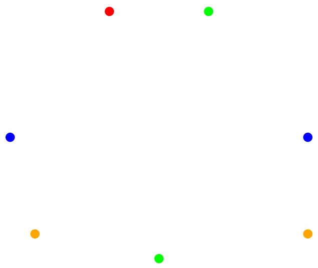

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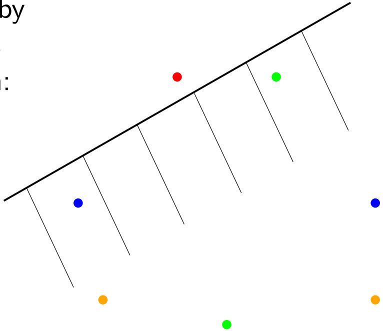

Translates

Conjecture (Pach '80)

For every planar convex set D and k there is an m such that we can color any finite $X \subset \mathbb{R}^2$ with two colors such that every translate of D with at least m points contains both colors.

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Theorem (P. '13)

For every m there is a finite $X \subset \mathbb{R}^2$ such that for every two-coloring of X there is a unit disk that contains at least m points from X and all have the same color.

Proper four-coloring

Theorem (Cardinal-Korman '13)

For any planar convex set D, any finite $X \subset \mathbb{R}^2$ can be 4-colored such that any translate of D with at least 2 points is non-monochromatic.

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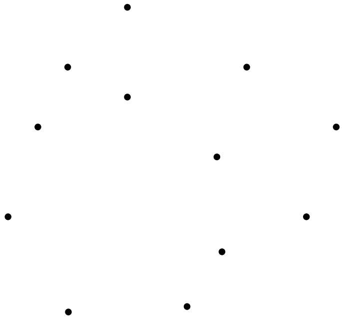

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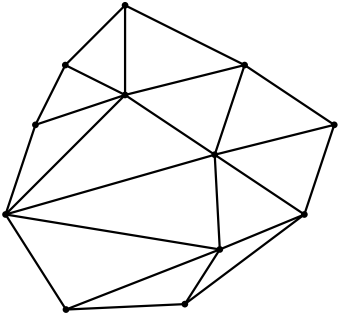

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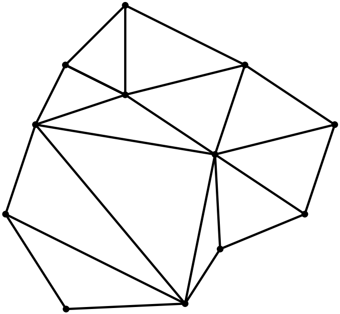

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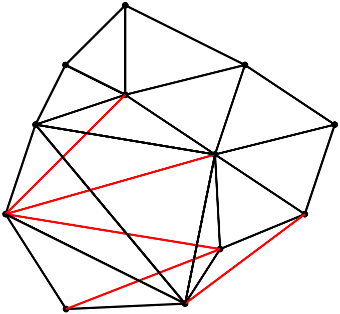

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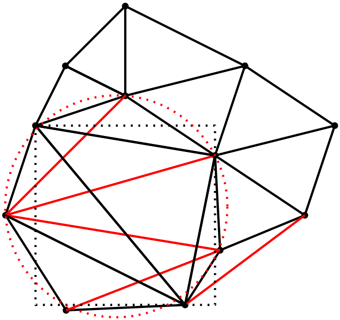

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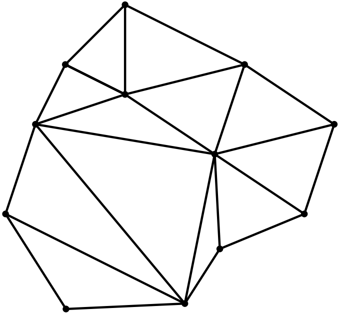

Lemma: Generalized Delaunay triangulation is planar graph.

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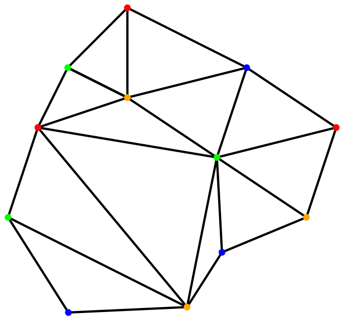

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We can apply the Four Color Theorem. □

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(ii) for every homothet D' if D' contains t points from R colored with the same color, D' also contains a point from $X \setminus R$ that has this same color.

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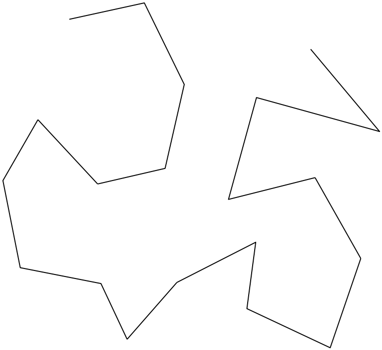

Take a monochromatic path.

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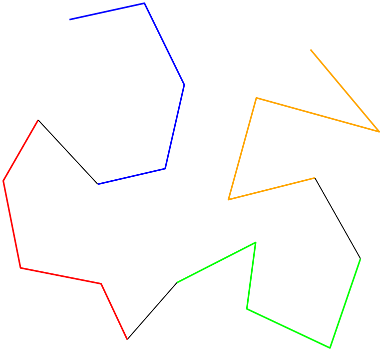

Divide it into sections of length $\approx t$.

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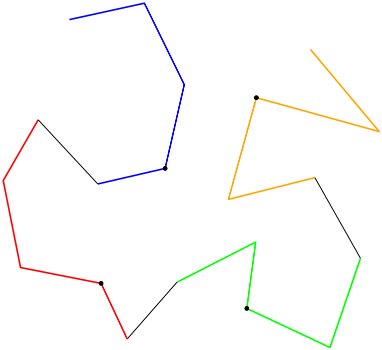

Put a non-D-extremal, non-end point from each to R.

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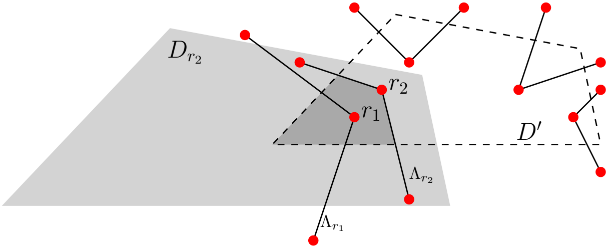

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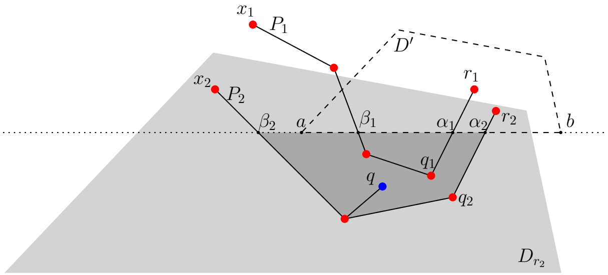
2. Sections of three $r \in R \cap D'$ intersect D' in same side. ↯ □



Summary of new results and related questions

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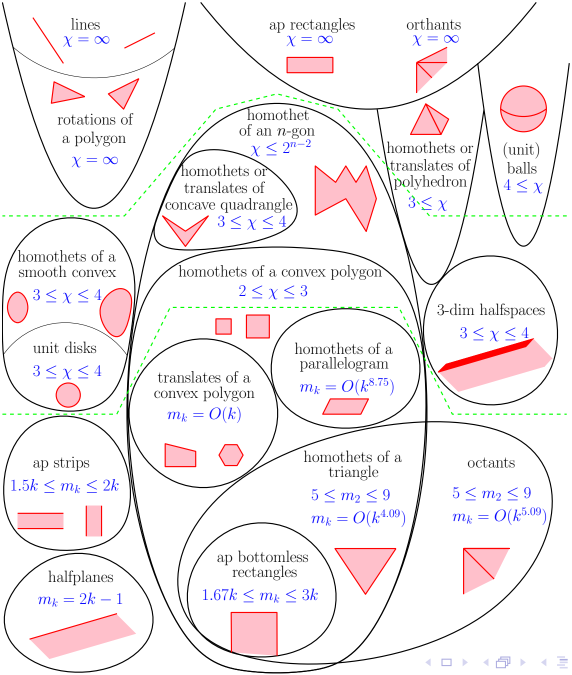
Summary of known results



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