Radon numbers grow linearly

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Radon's theorem

Theorem (Radon '21)

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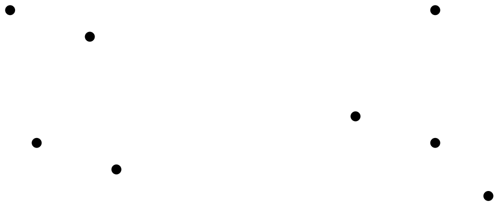

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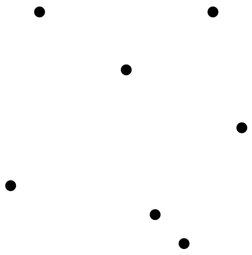

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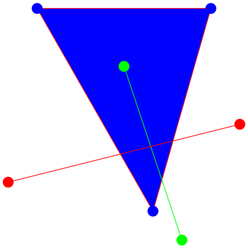

Convexity spaces

Definition (Convexity space)

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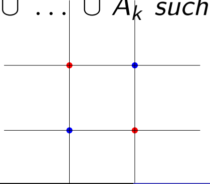
Note that for $(d+1)(k-1)+1$ points this is not always true.



Here $d = k = 2$ but there is no partition, so even Radon fails.

Eckhoff's conjecture

Question (Calder '71, Eckhoff '78)

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Theorem (Doignon '75, Onn '91)

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$\frac{5}{4}2^d + 1 \leq r_2(\mathbb{Z}^d) \leq d2^{d-1}+3.$

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So $r_k(\mathbb{Z}^d) \leq (r(\mathbb{Z}^d)-1)(k-1)+1$ seems open.

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Remark: $\beta = \Omega(\alpha^{r^f})$.

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This gives β-dense s-uniform hypergraph $\mathcal{H}$ on $t = 100ks$ vertices.

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Thus there are k intersecting vertex-disjoint convex hulls. □

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Constructions like Bukh's showing $r_k > (r-1)(k-1)+1$?

Open

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Thank you for your attention!