Finding the biggest and smallest element with one lie

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In our model - Fixed number of lies, best worst-case algorithm, questions can depend on previous answers.

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Finding the biggest and the smallest with k lies - Our topic.

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championship graph - digraph, ab is an edge iff a has beaten b

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After this we "change our mind" and reply such that the team with in-degree k always wins.

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Adversary can answer such that only matches between two newbies result in loss of two pebbles $\Rightarrow \frac{3}{2}n - 2$ questions are needed.

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How many comparisons are needed?

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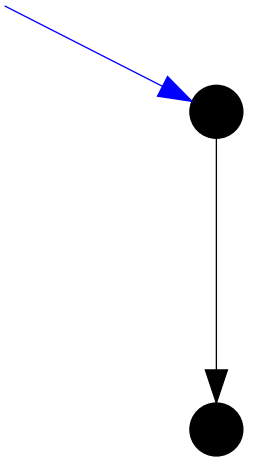

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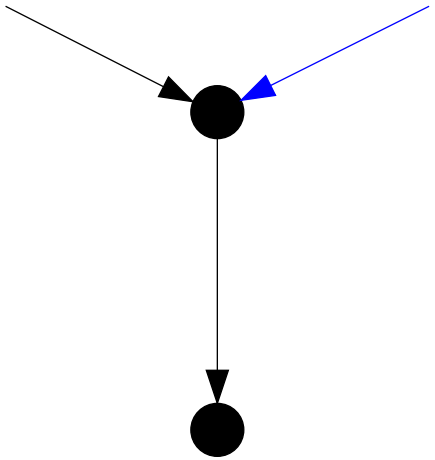

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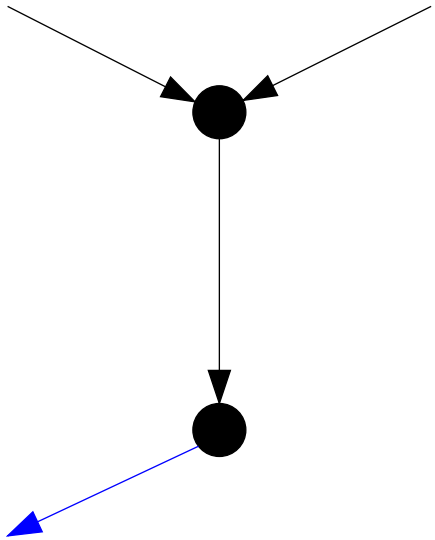

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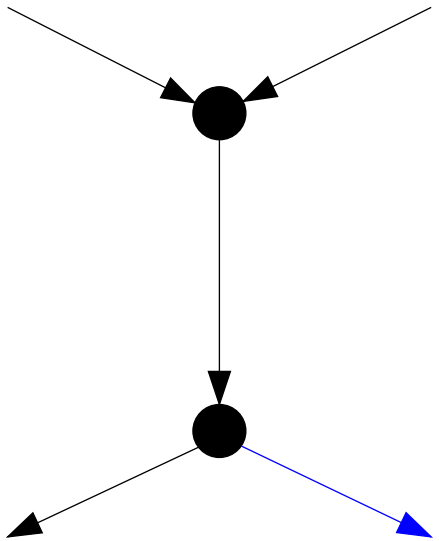

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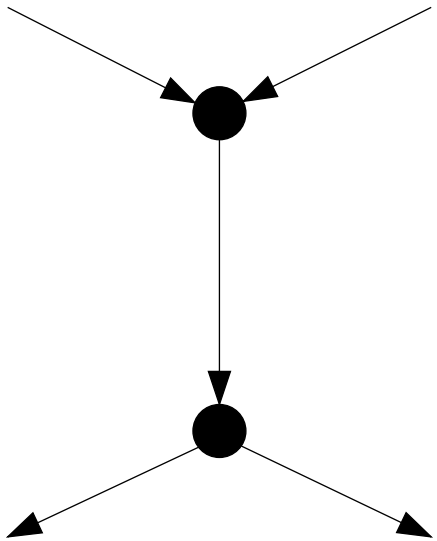

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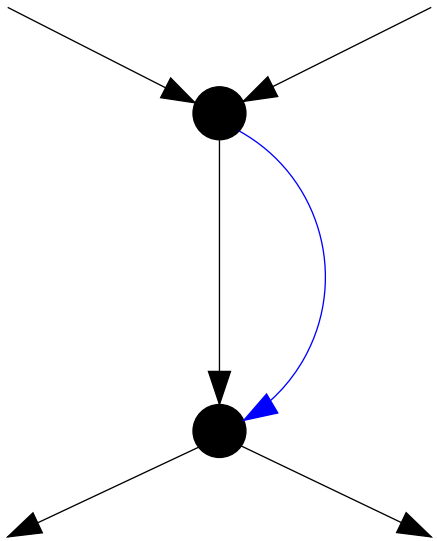

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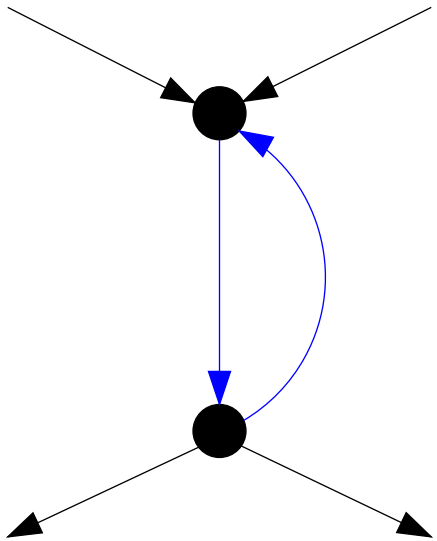

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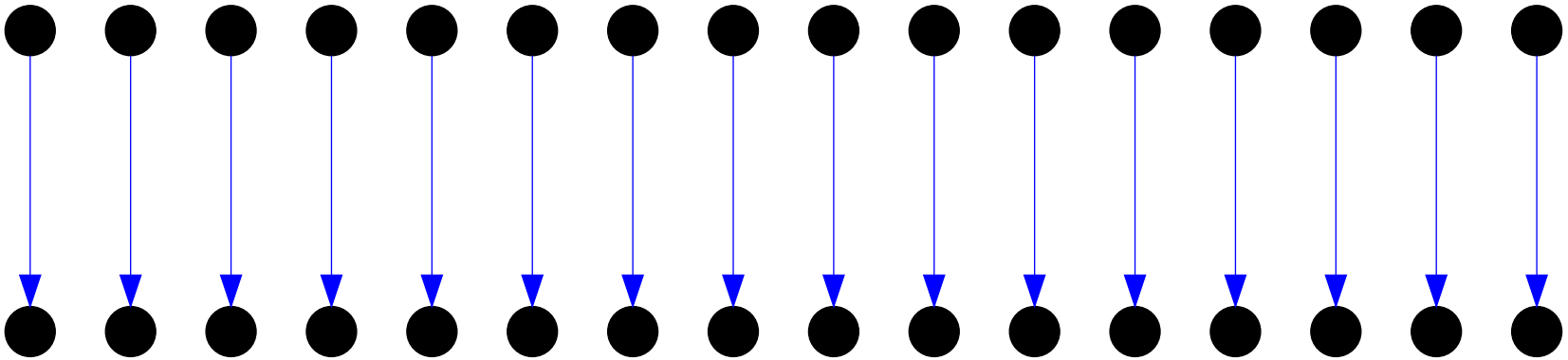

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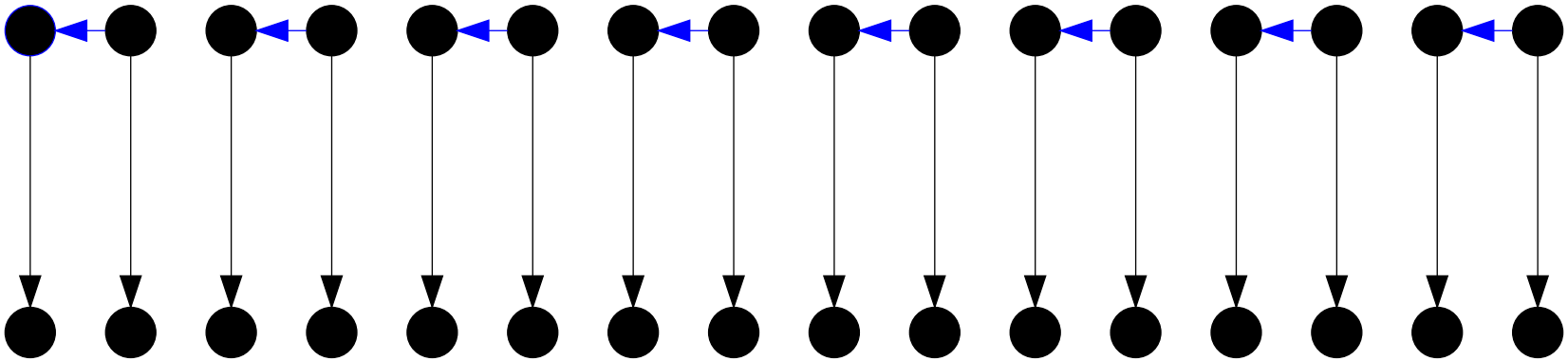

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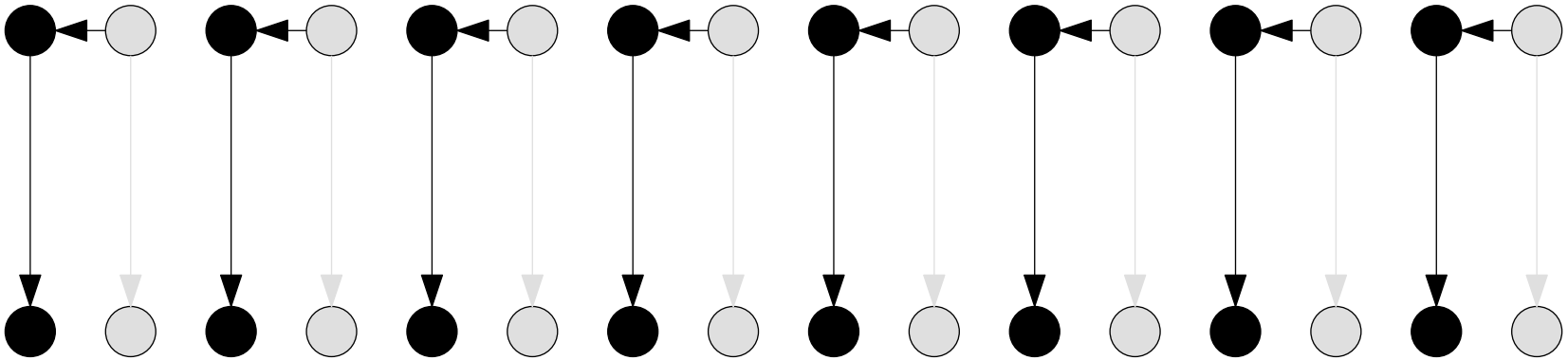

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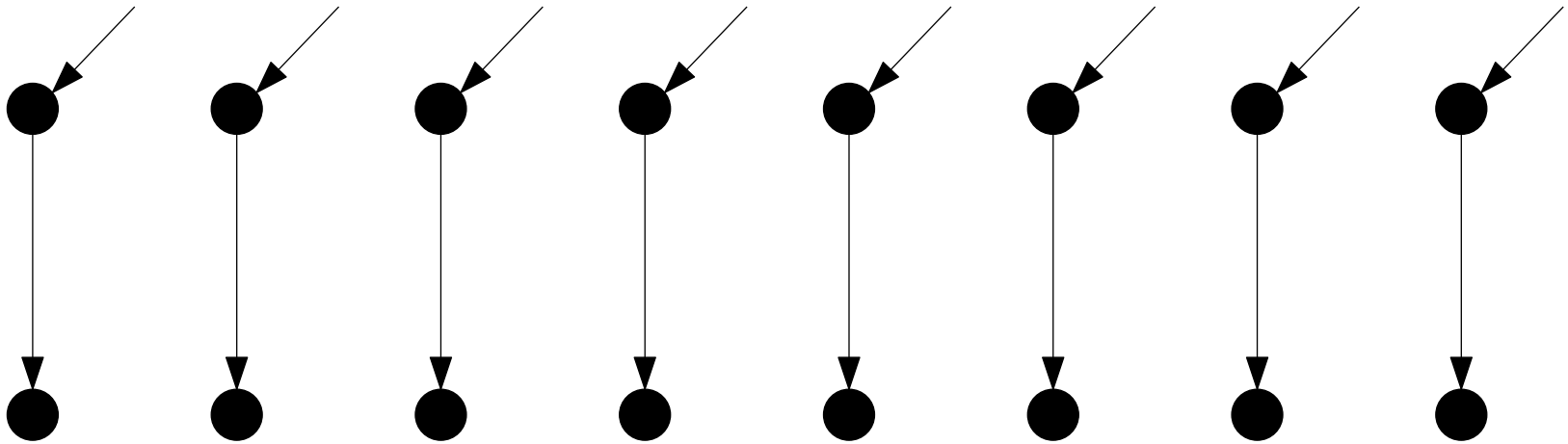

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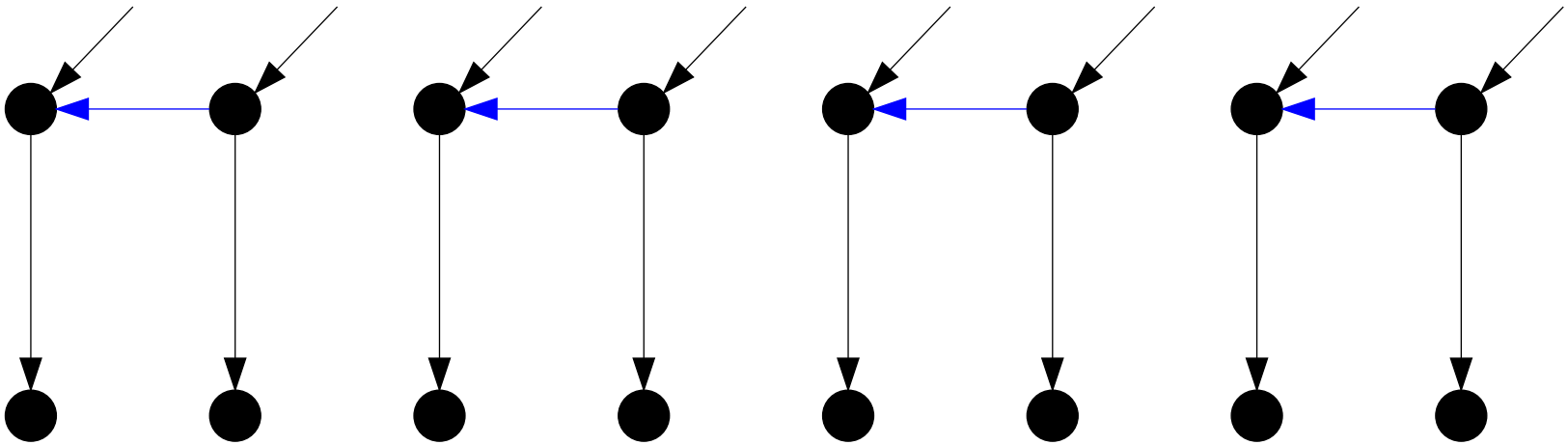

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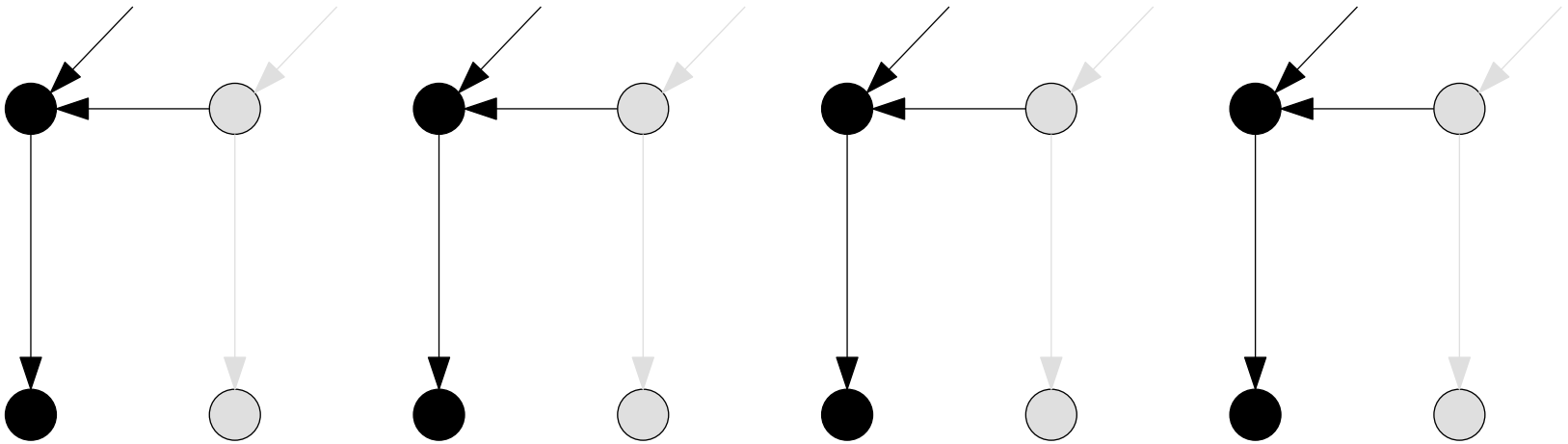

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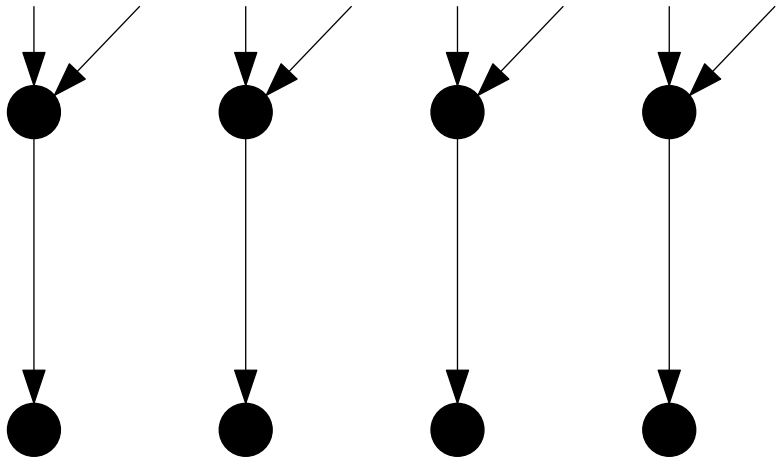

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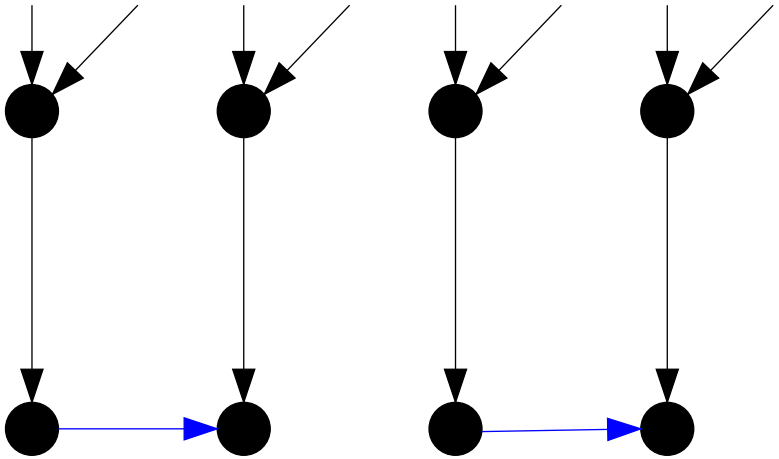

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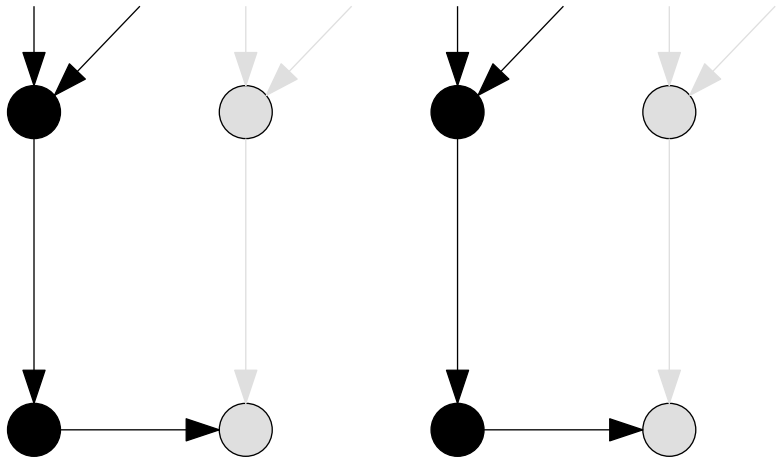

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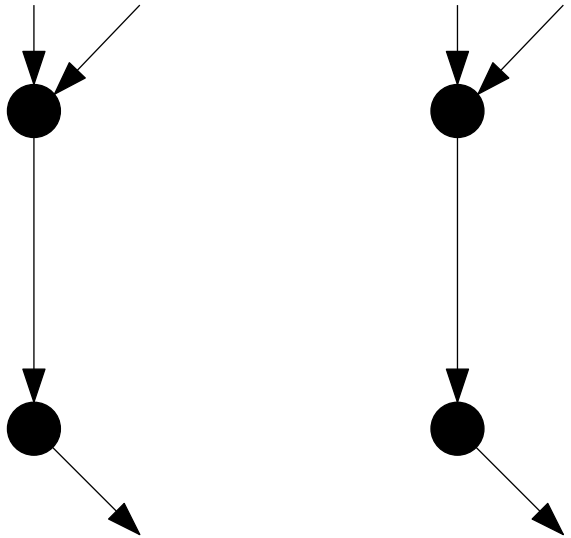

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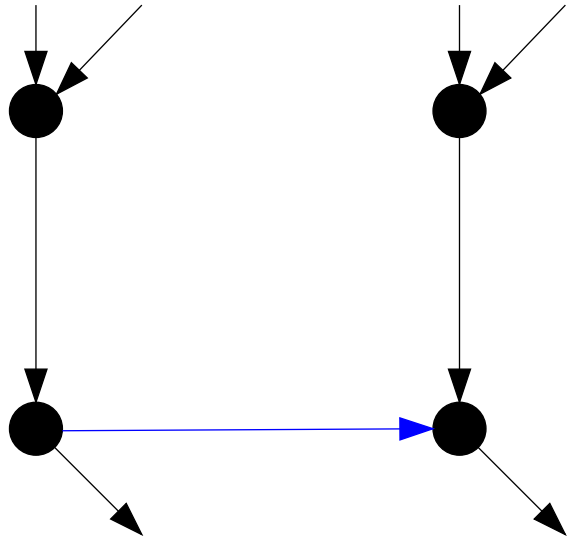

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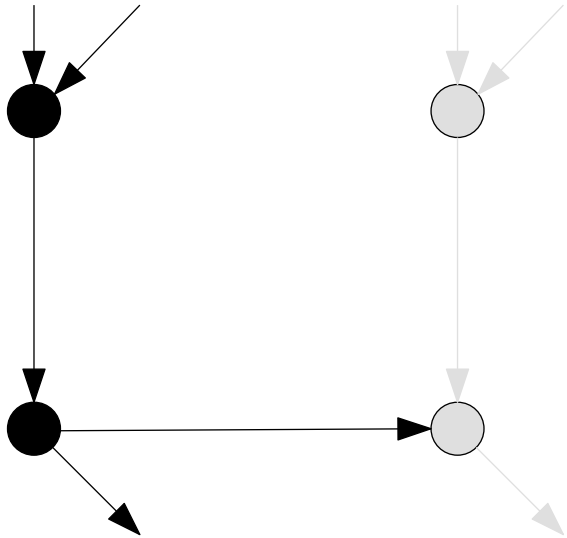

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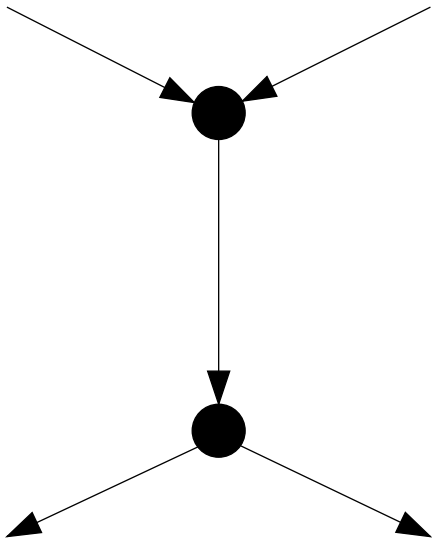

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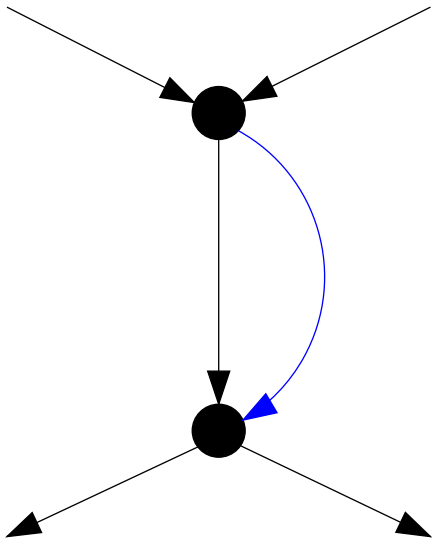

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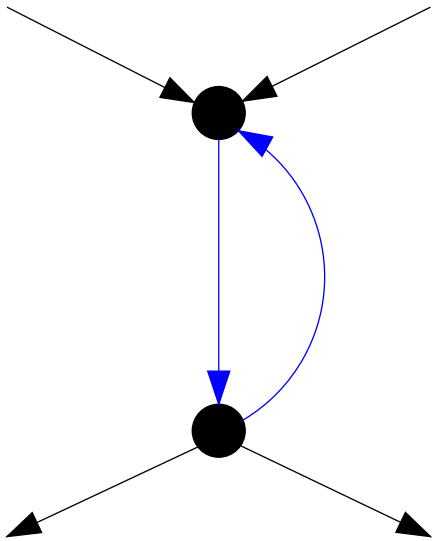

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If later x looses twice and y wins twice, then $c(x,y) = 1$.

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Proof is by analyzing a lot of cases.

Thank you for your attention!