Drawing cubic graphs with the four basic slopes

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Graph Drawing '11 - Eindhoven

Definition

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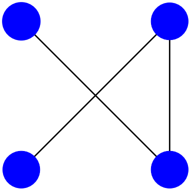
Example



A straight-line drawing of P_3.

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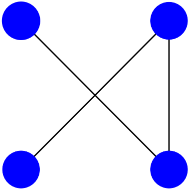

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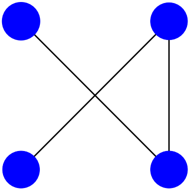

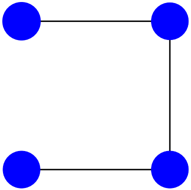

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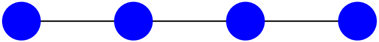

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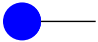

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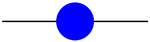

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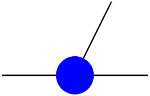

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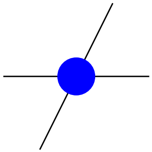

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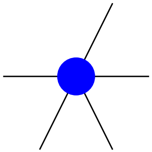

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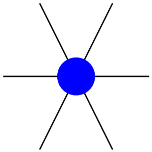

Question: Bounding slope number from above by a function of the maximum degree?

Earlier results

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Graphs with maximum degree 5 *can have arbitrarily large slope number.*

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Question: Is the slope number of quartic graphs bounded?

Our goal is to give an optimal answer for cubic graphs.

Reminder

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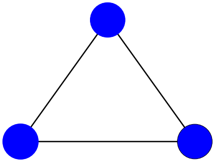

The slope number of K_3 is three.

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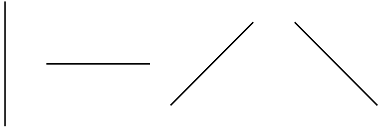

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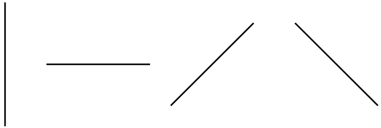

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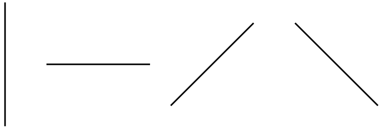

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Note that all previous theorems about cubic graphs used unnatural slopes.

Subcubic Lemma

Theorem (Keszegh, Pach, P., Tóth)

Let G be a connected graph whose every vertex has degree at most three and has at least one vertex with degree less than three. Then G has a straight-line drawing with the four basic directions.

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- *The neighbor of degree three vertex of the removed cycle need not have degree three.*

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- *The x-coordinates of all the vertices are a linear combination with rational coefficients of $x_1, \ldots, x_n$.*

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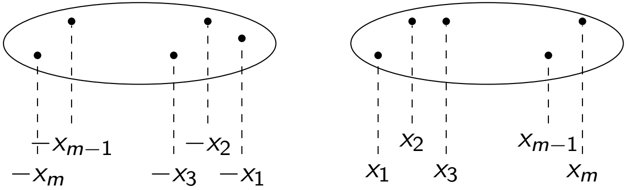

Figure: The x-coordinates of the degree 2 vertices is suitably chosen and one component is rotated and translated to make the M-cut vertical.

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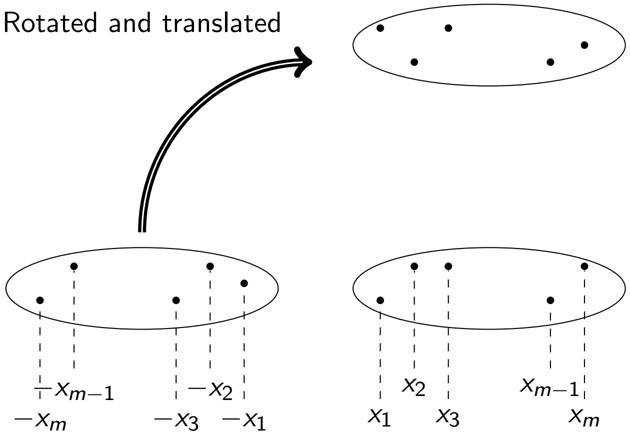

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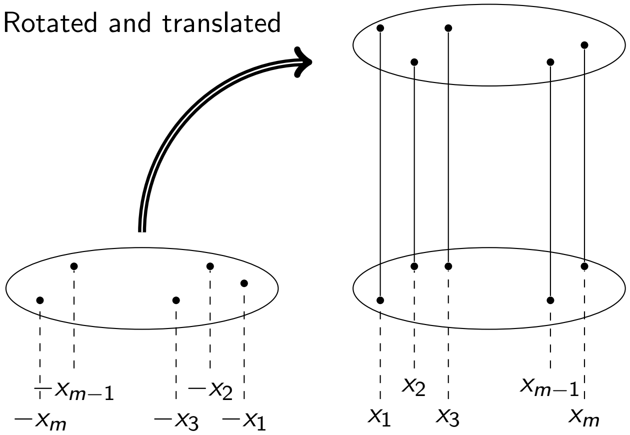

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Corollary

Cubic graphs that cannot be drawn with the four basic slopes must be three connected.

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Remark

Intuition: For a connected cubic graph G on n vertices, if there is a component C, not a cycle, on s vertices with $s << n$ such that the degree of all vertices in C is at least 2, then G has a M-cut.

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Lemma

Every connected cubic graph on n vertices contains a cycle of length at most $2\lceil \log(\frac{n}{3}+1) \rceil$.

Finding a short cycle

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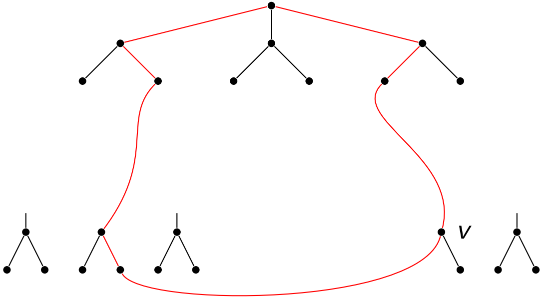

Figure: Finding a cycle in the BFS tree using that the left child of v already occurred.

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Lemma (Intuition in numbers)

Every connected cubic graph on $n > 2s - 2$ vertices with a supercycle with s vertices contains a suitable M-cut of size at most $s - 2$.

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The only graphs that requires to be checked are Petersen and Tietze's graph.

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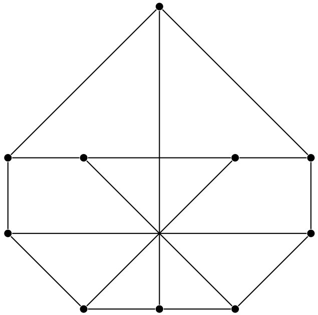

Figure: The Petersen graph drawn with the four basic slopes.

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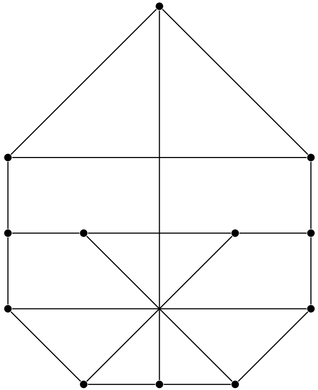

Figure: The Tietze's graph drawn with the four basic slopes.

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- None of the slopes in S can occur three times in the drawing of K_4.
- Two slopes, say s_1 and s_2, occur twice.
- These slopes must form a parallelogram that can be transformed to a square with an affine transformation.

□

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What about higher maximum degree and other graph classes?

Thank you for your attention!