Drawing cubic graphs with at most five slopes

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COMBSTRU - Barcelona, 2006

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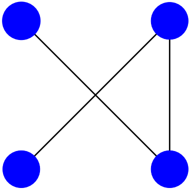

A straight-line drawing of P_3.

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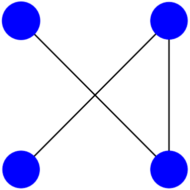

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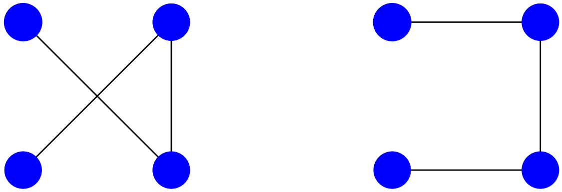

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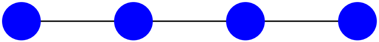

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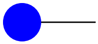

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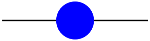

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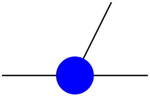

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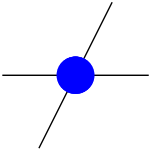

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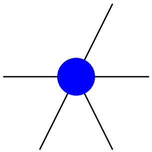

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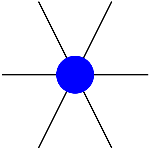

Question: Bounding slope number from above by a function of the maximum degree?

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Even for graphs with maximum degree five, the slope number can be arbitrarily large.

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The case of maximum degree four remains open.

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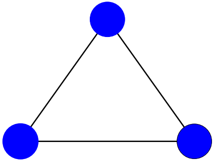

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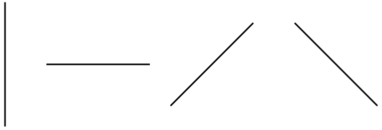

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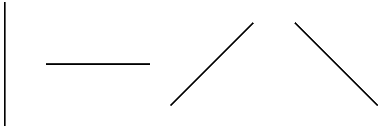

Theorem

Let G be a connected graph whose every vertex has degree at most three and has at least one vertex with degree less than three, then G has a straight-line drawing with the four basic directions, thus the slope number of G is at most four.

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There are some special conditions that are maintained during the induction, these will be mentioned later.

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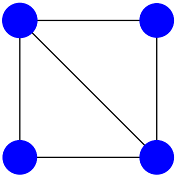

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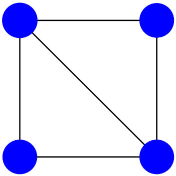
And v can be added easily to this drawing, since its neighbor (eg., the upper-right vertex) can have only two edges in G'.



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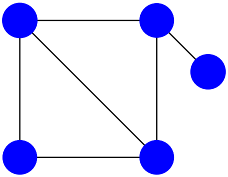

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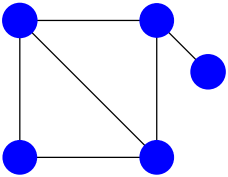

Note that if v is placed *close enough* to its neighbor, it cannot cause any troubles.

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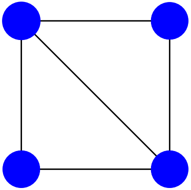

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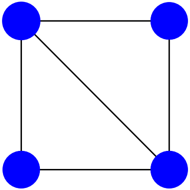

Later the proof will use that G has no Θ-subgraph, so if G has one, it has to be eliminated using induction.

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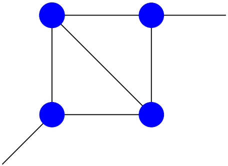

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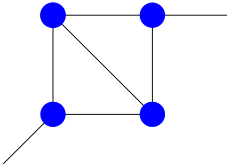

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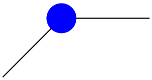

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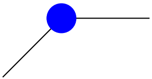

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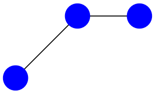

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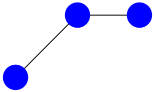

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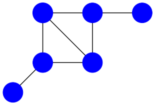

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(Everything is ok if the Θ-subgraph is small enough.)

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So from now on the following propositon can be supposed:

Proposition

The degree of each vertex is at least two, there are no Θ*-subgraphs and* G *is two-connected.*

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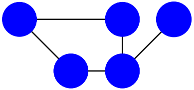

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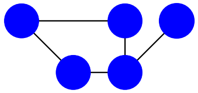

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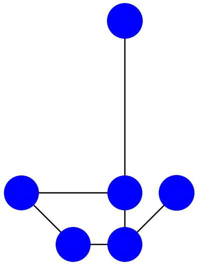

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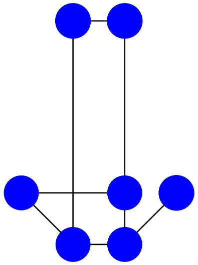

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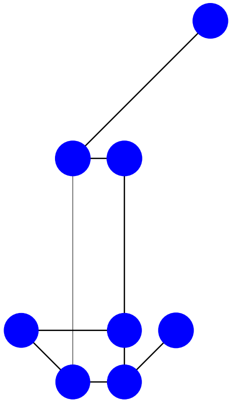

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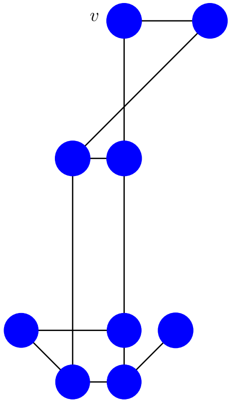

The first question is, what happens if a vertex from G' has more neighbors from C.

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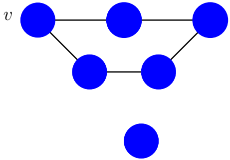

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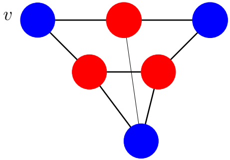

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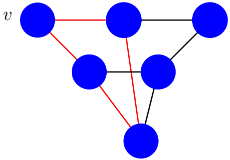

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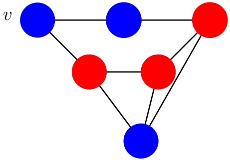

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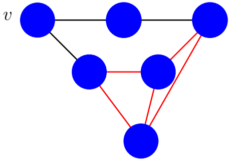

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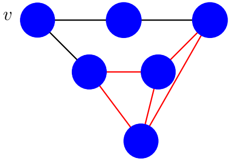

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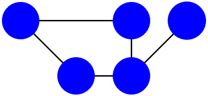

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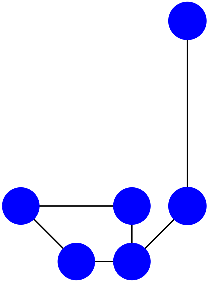

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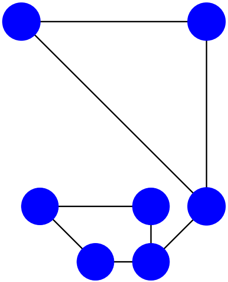

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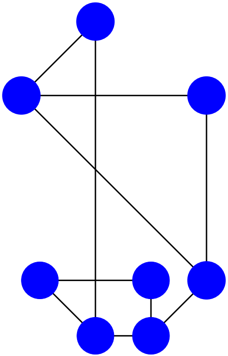

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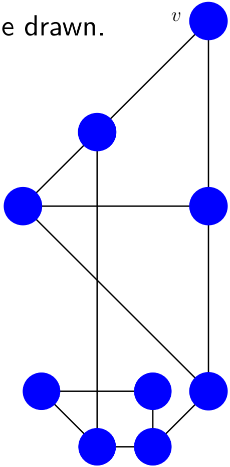

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Let G be a connected graph whose every vertex has degree at most three and has at least one vertex with degree less than three, then G has a straight-line drawing with the four basic directions, thus the slope number of G is at most four.

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The main idea of the proof is very simple:
One edge is deleted and the rest of the graph is drawn using the first theorem.

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Delete an arbitrary edge from each component, draw the rest of the graph, then put back the edges.

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Thank you for your attention!