The ratio of the surface-area and volume of finite union of copies of a fixed set in $\mathbf{R}^n$

Thesis

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1 Introduction

1.1 Motivation

In a paper of Tamás Keleti [1] the following question emerged which can also be found at Marianna Csörnyei's collection of unsolved problems (http://www.home pages.ucl.ac.uk/∼ucahmcs/):

Conjecture 1.1. *If H is a finite union of unit squares, then the ratio of the perimeter and area of H cannot exceed 4.*

Note that the bound 4 can be reached easily: take only one square. In [1] it was proved that there exists a constant C such that if H is a finite union of unit squares, then the ratio of the perimeter and area of H cannot exceed C. So far the best known result for C had at least 3 digits, far from the conjectured constant 4. We sharpen this result by showing that $C = \frac{2\pi}{\frac{1}{2}\log 2 + \frac{\pi}{4}} \sim 5.551$ is a good constant.

We remark that the problem somewhat resembles to the isoperimetric inequality (cf. [3]) and the Hadwiger-Kneser-Poulsen conjecture (cf. [2]). This latter states the following: if $P_1, \dots, P_k$ and $Q_1, \dots, Q_k$ are points in $\mathbf{R}^n$ such that $d(P_i, P_j) \leq d(Q_i, Q_j)$ for every $1 \leq i < j \leq k$ then $v\Big(\cup_{i=1}^k B(P_i, 1)\Big) \leq v(\Big(\cup_{i=1}^k B(Q_i, 1)\Big)$ where $d(P, Q)$ denotes the distance of points P and Q, $v(H)$ the volume of H and $B(P, r)$ the ball with centre P and radius r. We used some of the ideas that can be found in the proofs of results related to these problems.

1.2 Organisation

The thesis is organised as follows.

In section 2 the **thickness function** will be defined. We will prove some basic properties of it that will be useful in section 3. The **thickness** also plays crucial role in the formulation of Theorem 3.1. This theorem will give a bound for the ratio of the surface-area and volume when we take a finite union of reasonably nice (not necessarily congruent) sets. The proof of the theorem will be based on a method we call stripping. However the proof is 'almost' elementary, the basic idea of it gave rise to the interesting vector-field of the average radius-vectors in section 4.

In section 4 Theorem 3.1 will be proved using advanced tools. The proof will give a generalisation of Theorem 3.1. We will define a vector-field the elements of which we call **average radius-vectors**. The definition is in close connection with the definition of the **average thickness**. The properties of this vector-field turned out to be very interesting: e.g. it is smooth in cases when one wouldn't expect it to be smooth (this has some analogy with the convolution). In this section we will use some terminology and methods from differential topology (cf. [4]). In section 5 we will apply the main theorems of section 3 and 4 in the case of squares and circles.

In section 6 we will consider the ratio of the surface-area and volume in the case when the finite union consists only of translated copies of a fixed set. Sharp results will be obtained in the case of triangles, parallelograms and regular polygons.

In section 7 we prove that Conjecture 1.1 is true if we also assume that the union has the property that every point in the plane can belong to the interior of at most two squares. The Lemma in this section is somewhat similar to the isoperimetric inequality (cf. [3]). Finally section 8 is a collection of examples and counterexamples that were found to falsify 'naive' conjectures that emerged during the work.

1.3 Acknowledgements

I would like to express my gratitude to Tamás Keleti for his very useful help. Section 4 is completely due to discussions with him, and these discussions affected the structure and results of all the section. I also would like to thank Dömötör Pálvölgyi for the discussions we had. The results based on his ideas can be found in section 5 and 6.

2 Thickness

2.1 Some definitions

Definition 2.1. *Let H be a compact set that is a closure of an open set in $\mathbf{R}^n$ and P is a point on its boundary with the property that it has a tangent hyper-plane (we will refer to such points as 'smooth point'). First we define the* ***thickness function of the set*** H ***at point*** P *as*

$$t_{P,H}(x) := l_{P,H}(x) \sin \Phi_{P,H}(x) \quad (x \in S^{n-1})$$

where $l_{P,H}(x)$ denotes the length of the longest (oriented and closed) line segment in H starting from point P having direction x and $\Phi_{P,H}(x)$ is the angle between vector x and the tangent hyper-plane at P.
After that we define the ***average thickness of the set*** H ***at point*** P ***with respect to*** μ *as*

$$T_{P,H,\mu} := \int_{S^{n-1}} t_{P,H}(x) d\mu(x)$$

where $\mu(x)$ is any probability measure on the Borel subsets of S^{n-1}.

Definition 2.2. *The* ***thinness*** *of a set (of the same type as above) H with respect to μ is defined as*

$$T_{H,\mu} := inf\{T_{P,H,\mu} : P \in \partial H\}$$

and the ***slimness*** *is defined as*

$$S_H := sup\{T_{H,\mu} : \mu \in \boldsymbol{P}(S^{n-1})\}$$

i.e. the 'largest' possible thinness of a set, where $\boldsymbol{P}(S^{n-1})$ denotes the Borel probability measures on S^{n-1}.

Remark 2.3. *It is clear that if $P \in \partial H_1$ and $P \in \partial H_2$, a smooth point of both boundaries with the same tangent hyper-plane, then if $H_1 \subset H_2$, then $t_{H_1,P}(x) \leq t_{H_2,P}(x)$ and so $T_{H_1,P,\mu} \leq T_{H_2,P,\mu}$.*

2.2 Some properties of the average thickness

We start with some notations.

Definition 2.4. *If H is a fix polyhedron, Let K_H denote the $n-2$-skeleton of H. For $P \in \mathrm{int}H$ let $K_H(P)$ denote the subset of S^{n-1} we get the following way: take the unit sphere around P and intersect it with all the lines connecting P with the points of K_H.*

Lemma 2.5. *If H is a polyhedron, F is a face of it and $\mu \in \boldsymbol{P}(S^{n-1})$ is 0 on the intersections of S^{n-1} and any hyper-plane in $\mathbf{R}^n$ (i.e on the 'spherical spheres'), then the average thickness $T_{P,H,\mu}$ is continuous on* intF *as the function of P (considering F as a topological subspace of the hyper-plane it spans).*

Proof of Lemma 2.5. By assumption $\mu(K_H(P)) = 0$. Now fix an ε between 0 and $\frac{\pi}{2}$. For this ε there exist a δ such that if the distance of P and Q (it will be denoted by $d(P,Q)$ in the future) is less than δ, then for any point $R \in \partial H$ the angle PRQ is at most ε: indeed, the points X for which the angle PXQ is at least ε are contained in a sphere with centre $M(P,Q)$ ($M(P,Q)$ is the midpoint of the line segment PQ) and radius $\frac{d(P,Q)}{2\mathrm{tg}(\frac{\varepsilon}{2})}$ so it is in a sphere with centre P and radius $\frac{d(P,Q)}{\mathrm{tg}(\frac{\varepsilon}{2})}$. So it is enough to choose δ such small that $d(P,K) > \frac{\delta}{\mathrm{tg}(\frac{\varepsilon}{2})}$ which is clearly possible ($d(P,K_H) > 0$ because K_H is a compact set and $P \notin K_H$).

Let's take a direction disjoint to the ε neighbourhood of $K_H(P)$ (we will denote it by $K_H(P)_\varepsilon$). All the longest segments started from the points of the δ neighbourhood of P in this direction have the property that their endpoint are not in K_H which means they are in the interior of a face of the polyhedron (interior means the same as in the claim). This is true: we have chosen δ this way, i.e. such that if $d(P,Q) < \delta$, then $K_H(Q) \subset K_H(P)_\varepsilon$. This also shows that $t_{Q,H}(x)$ is continuous even as the function of $(Q,x) \in \overline{P_\varepsilon} \times \left(S^{n-1} - K_H(P)_\varepsilon\right)$. But this set is compact, so it is uniformly continuous on this set. So for any $\varepsilon' > 0$ there exists a $\delta'(< \delta)$ such that if $d(P,Q) < \delta'$ then $|t_{P,H}(x) - t_{Q,H}(x)| < \varepsilon'$.

Now it is easy to prove the lemma. If $d(P,Q) < \delta'$ then

$$|T_{P,H,\mu} - T_{Q,H,\mu}| \leq \int_{S^{n-1}} |t_{P,H}(x) - t_{Q,H}(x)| d\mu(x) =$$

$$= \int_{S^{n-1}-K_H(P)_\varepsilon} |t_{P,H}(x) - t_{Q,H}(x)| d\mu(x) + \int_{K_H(P)_\varepsilon} |t_{P,H}(x) - t_{Q,H}(x)| d\mu(x) \leq$$

$$\leq \varepsilon' + \mu\Big(K_H(P)_\varepsilon\Big) 2D < 2\varepsilon'$$

where D is the diameter of H and ε is chosen such that $\mu(K_H(P)_\varepsilon) < \frac{\varepsilon'}{2D}$. It is possible, because $\mu(K_H(P)) = 0$ so $\lim\limits_{\varepsilon \to 0} \mu(K_H(P)_\varepsilon) = 0$ because $\cap_{\varepsilon>0} K_H(P)_\varepsilon = K_H(P)$ ($K_H(P)$ being compact). This finishes the proof. □

We can see that in the proof we only used that $\mu(K_H(P)) = 0$. That means we also proved the following:

Corollary 2.6. *If H is a polyhedron, and Q is a point of the interior of face F and $\mu(K_H(Q)) = 0$ then the average thickness $T_{P,H,\mu}$ is continuous in point Q as the function of P.*

We are ready to prove the main theorem in this section:

Theorem 2.1. *If H is a polyhedron, F is a face of it and $\mu \in \boldsymbol{P}(S^{n-1})$, then the set of points in F where $T_{P,H,\mu}$ is not continuous is a Lebesgue-zero set.*

Proof of Theorem 2.1. From corollary 2.6 we know that $T_{P,H,\mu}$ is continuous at all the points where $\mu(K_H(P)) = 0$. So it is enough to prove that $\lambda\Big(\{P : \mu(K_H(P)) > 0\}\Big) = 0$. But this is easy: we prove that there are countably many hyper-planes on the face F the union of which contains all the P with $\mu\Big(K_H(P)\Big) > 0$. Indeed, for every point P there exists a $n-2$-dimensional polyhedron in the $n-2$ skeleton of H such that connecting only the points of this with P and taking the intersection with the unit sphere yields a set with positive μ measure. Denote this polyhedron by J. Now if the line defined by P and Q is not parallel with the affine-span of this polyhedron, then the resulting sets on the sphere are disjoint, which means that there can be only countably many hyper-planes in F parallel with the affine-span of J and containing a point P for which the intersection of the unit sphere around P and the lines connecting P with J has positive μ-measure. And H has finitely many $n-2$ dimensional polyhedra in its $n-2$ skeleton, which finishes the proof. □

Corollary 2.7. *$T_{P,H,\mu}$ is measurable.*

Note that this is also the consequence of the theorem of Fubini applied to $t_{P,H})(x)$ as a function both of P and x. However, with the use of the theorem of Fubini we will give an advanced proof in section 4.

3 Stripping

3.1 The main theorem

With a method we call stripping we will prove the following theorem.

Theorem 3.1. *If H is the union of a finite set $\mathcal{H}$ of polyhedra in $\mathbf{R}^n$ then for any fixed μ probabilistic measure the ratio of its surface-area and volume cannot be more than $\frac{1}{T_\mu}$, where T_μ is the infimum of the set $\{T_{A,\mu} : A \in \mathcal{H}\}$*

Proof of Theorem 3.1 We will prove this theorem in the plane. It is easier to carry the proof out in this case but it can be generalised without difficulty in higher dimensions.

We will use a method we call stripping, which is the following: Let's take all the sides of the boundary of the union (in the proof they will be divided into even smaller parts that will be called 'sides' again). We will place strips on these sides (which are naturally parts of the original sides of the the polygons in the union). A strip on a side is an intersection of a set bounded by two parallel half-lines, the endpoints of the side being the starting point of half-line with the polygon in the union to which the side belongs. The direction of a strip is the direction of he half-lines. In the plane it

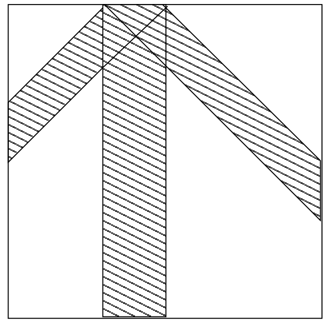

Figure 1:

can be simply represented with the angle α ($0 \leq \alpha < 2\pi$) these half-lines have with a given axis x. Figure 1. shows an example of 3 strips in a square. Let's take strips at angles $0 \leq \alpha_1 < \cdots < \alpha_n < 2\pi$ for all sides of the approximating polygon and positive weights $w_1, \dots w_n$ such that $\sum w_i = 1$. Let's denote the sides with $s_1, \dots s_k$ and let $a_{i,j}$ denote the area of the strip $s_{i,j}$ on the side s_i at angle α_j. Then we claim that in the case of polygons

$$\sum_i \sum_j w_j a_{i,j} \leq a(H)$$

where $a(H)$ denotes the area of H. Indeed, it is enough to prove this for rational weights. By multiplying with the common denominator, we have to prove the following: $\sum \sum w_j a_{i,j} \leq a(H) \sum w_i$ where the weights w_i are integers now. This is the same as the case when all the weights are ones but we can take an angle several times. From this point of view $N := \sum w_i$ is the number of strips. Let $0 \leq \beta_1 \leq \beta_2 \cdots \leq \beta_N < 2\pi$ denote the new angles of the strips (so now an angle can occur several times).

Now we claim that for a fixed k ($1 \leq k \leq N$) every point can belong to at most one strip of angle α_k. Indeed, if a point P belongs to two strips, take the two corresponding sides. There are two unique points Q and R on the sides such that the angle of the half lines QP and RP with the axis x is β_k. So the points P, Q and R are on the same line such that Q and R are on the same side of P. Let's suppose that Q is the one nearer to P. Now the contradiction is clear, because then the strip belonging to the side containing point R covers point Q, so Q cannot belong to the boundary (all strips being part of the union).

Let's suppose that the measure μ has finite support consisting of angles $0 \leq \alpha_1 < \cdots < \alpha_n < 2\pi$. Let $\mu(\alpha_i) = w_i$ so that $w_i > 0$ (and obviously $\sum w_i = 1$). Let's take the sides in the stripping such that the area of each strip $s_{i,j}$ will be $|s_i| t_{M_i,H_i}(\alpha_j)$ where $|s_i|$ is the length of the side s_i, M_i its midpoint and H_i is the polygon corresponding to the side s_i (we used some of the notations of the definition). We remark that H_i can be the same polygon for several values of i.

Such choice of the sides is possible. Indeed, if we can choose the sides in a way that all the strips will be trapezoids then the areas will be exactly what we claimed. But that is easy: project all the vertices of a fixed polygon in all the directions $\alpha_1 + \pi$, $\alpha_2 + \pi$, $\dots \alpha_n + \pi$ on all of its sides (in some cases we will not get a point). After this if a part of a side lies on the boundary of the union let's subdivide it with all the points we got from the previous projections. Now all the strips are trapezoids on these smaller sides: indeed, if there is a strip which is not of this shape, it would contain a vertex not lying on the two parallel lines that bounds the strip. But in this case the side to which the strip is associated would contain inside of it the projection of the vertex in direction $\alpha_k + \pi$ (where α_k is the angle of the strip we considered). That is a contradiction. Figure 2. shows some of the final strips in a union of squares.

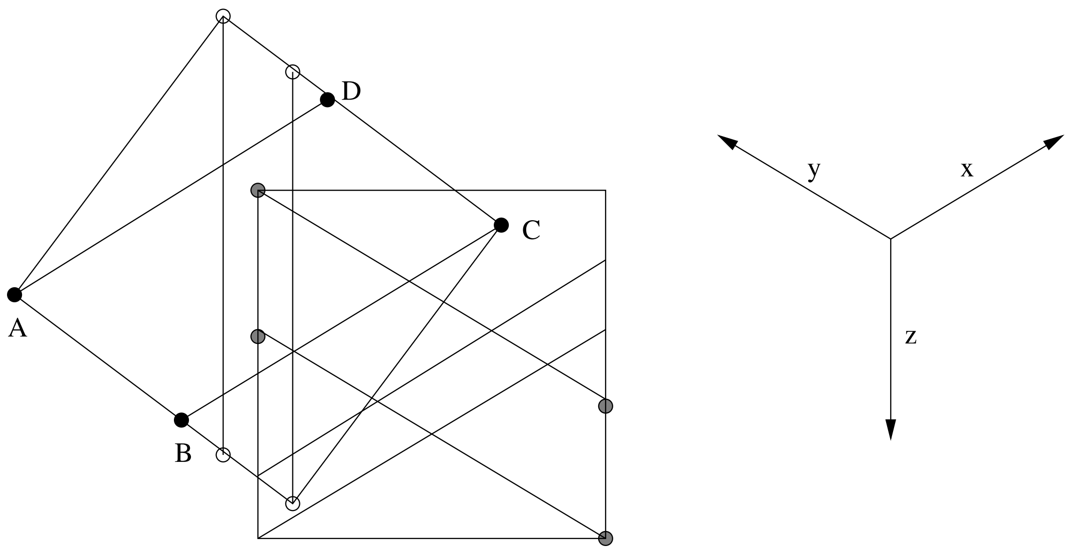

Figure 2:

Let's consider a strip on the side s_i at direction α_j. It is a trapezoid, so its area is the product of the length of its mid-line and its height. Its mid-line's length is $l_{M_i,H_i}(\alpha_j)$ and its height is $|s_i| \sin \Phi_{M_i,H_i}(\alpha_j)$ by the definition of Φ_{M_i,H_i} so their product is

$$l_{M_i,H_i}(\alpha_j)|s_i| \sin \Phi_{M_i,H_i}(\alpha_j) = |s_i| t_{M_i,H_i}(\alpha_j)$$

by the definition of the thickness function.

Now by summing the areas of the strips with weights w_i we can get at most $a(H)$. On the other hand if we sum first in j, then in i and use the definition of T_μ we get that

$$\sum_i |s_i| \sum_j w_j t_{M_i,H_i}(\alpha_j) = \sum_i |s_i| \int t_{M_i,H_i} d\mu =$$

$$= \sum_i |s_i| T_{M_i,H_i,\mu} \geq \sum_i |s_i| T_\mu = p(H) T_\mu$$

($p(H)$ denotes the perimeter of H). This shows that $p(H)T_\mu \leq a(H)$ so $\frac{p(H)}{a(H)} \leq \frac{1}{T_\mu}$ and that is what we wanted to prove. So we are finished if μ has finite support. Now let's proceed with the general case.

We know that there exists a a sequence μ_i of finite-support measures such that $\lim \int_0^{2\pi} f d\mu_i = \int_0^{2\pi} f d\mu$ for all functions f that are not continuous in finitely many points and even in those have left and right limits. This is easy: divide $[0, 2\pi)$ into

i equal intervals $\left[\frac{2\pi k}{i}, \frac{2\pi(k+1)}{i}\right)$ and let μ_i be on this interval the μ measure of it concentrated in its midpoint. The proof of the fact that this choice works is left to the reader. So $T_{P,H,\mu_i} \longrightarrow T_{P,H,\mu}$ for every P in the interior of a fixed side I, and then for every ε there is an $N(\varepsilon)$ such that $|T_{P,H,\mu_{N(\varepsilon)}} - T_{P,H,\mu}| < \varepsilon$ except for a set of Lebesgue measure less then ε (here we used the fact that $T_{P,H,\mu}$ is measurable!). Now in the stripping let's use intervals with midpoints in the set on which $|T_{P,H,\mu_{N(\varepsilon)}} - T_{P,H,\mu}| < \varepsilon$ holds. By the definition of the Lebesgue measure we can take finitely many intervals such that the measure of the set which they do not cover is less the ε. This way we get with a same argument like before that

$$a(H) \geq (p(H) - n\varepsilon)(T_\mu - \varepsilon)$$

where n denotes the number of sides of the union H. The second factor on the right side comes from the fact that on all the sides of the strips $T_{M,H,\mu_{N(\varepsilon)}} > T_{M,H,\mu} - \varepsilon \geq T_\mu - \varepsilon$ (M is the midpoint of the side). This gives the desired result by tending to 0 with ε. We finished the proof. □

After this theorem it is natural to investigate the thickness if there is a given set of polyhedra, or more generally, any subset of $\mathbf{R}^n$, from which we are allowed to choose sets in the union H. This will yield partial results when the bound 4 can be proved. We will see that result in section 5. We will also consider the problem of calculating the slimness in section 6.

We conclude this section with a remark: if μ has finite support, the values of μ at the points of the support will be called 'weights'.

3.2 The first application of Theorem 3.1

As the first application of Theorem 3.1 we prove a theorem analogous to Corollary 12. in [1]. We need a definition first.

Definition 3.1. *A set $H \in \mathbf{R}^n$ is called r-star shaped, if there exist a ball B in H with radius $r\,diamH$ such that $\forall x \in B \forall y \in H$ it is true that the line segment $[xy]$ is a subset of H.*

Theorem 3.2. *Let H be a finite union of r-star shaped polyhedra with diameter D in $\mathbf{R}^n$. Then the ratio of the surface-area and the volume of H cannot exceed $\frac{C(n,r)}{D}$ where $C(n,r)$ is a constant depending on n and r.*

Proof of Theorem 3.2 We can assume that D is 1 (by similarity). Take a point on the boundary of a polyhedron from the union. Then the polyhedron contains the convex hull of this point and the ball of radius r that is chosen according to the fact that the polyhedron is r-star shaped. This convex hull is called a drop in [1]. The distance between point P and the centre of the ball cannot be more than 1. Note that this drop is determined uniquely (up to congruence) with this distance (let's call it d) and radius r (and of course dimension n). It is easy to see that if in the vertex P of this drop we choose a hyper-plane such that the drop is contained in one of the half-spaces defined by this hyper-plane, then the definition of thickness of the drop makes sense in point P. If we choose the hyper-plane to be the affine span of the face (i.e tangent hyper-plane) of the polyhedron containing P, then the average thickness of the drop at point P is at most the average thickness of the polyhedron at P (cf. Remark 2.7). Finally it is easy too see that the average thickness of the drop with respect to the standard ('uniform') measure on S^{n-1} is the smallest when the hyper-plane is chosen such that it contains not only P but at least one more point from the boundary of the drop (i.e. when it indeed touches the drop). Let's denote this average thickness by $T(r,n,d)$ and we can observe that this is monotone decreasing as a function of d (r and n are fix). So the ratio of the surface-area and volume of H is bounded by $\frac{1}{T(r,n,1)}$, which finishes the proof. □

It is interesting that if we take a drop with radius r then the ratio of its surface-area and volume is depending only on r and n, but not on d: this is a corollary of Theorem 8.1. So if we would modify the definition of r-star shaped set such that instead of $r\text{diam}H$ we would take the radius of the ball to be simply r, we could except that the union of r-star shaped sets is bounded by a constant depending only on r and n. We do not know, if this is true or not.

4 An advanced proof of Theorem 3.1

4.1 The average radius-vector with respect to a random direction

In this section we give a more advanced proof for a slightly more general case using the theorem of Fubini and Gauss-Ostrogradski.

Theorem 4.1. *Let H be the finite union of sets in $\mathcal{H}$. About each element of $\mathcal{H}$ we assume that it is the closure of an open set in $\mathbf{R}^n$ with boundary that is the union of finitely many compact surfaces. We suppose that each of these surfaces can be parametrised by a map $f : U \longrightarrow \mathbf{R}^n$ where the compact $U \subset \mathbf{R}^{n-1}$ is the closure of an open set with Lebesgue-zero boundary and f is continuously differentiable on U. Let μ be a Borel probability measure that is zero on the main-spheres (the 'spherical hyper-planes') of S^{n-1}. In this case the ratio $\frac{s(H)}{v(H)}$ of the surface-area and volume of H cannot exceed $\frac{1}{T_\mu}$ where T_μ is the same as in Theorem 3.1.*

First we need a definition analogous to the definition of thickness.

Definition 4.1. *For a compact $H \subset \mathbf{R}^n$, $P \in H$ and $x \in S^{n-1}$ let $v_{x,H}(P)$ denote the longest vector $\overrightarrow{PQ}$ having direction x and the line segment PQ being the subset of H. Let's define the* ***a****verage radius-vector in point P* ***w****ith respect to μ as $\int_{S^{n-1}} v_{x,H}(P)d\mu(x)$ and denote it with $v_{P,H,\mu}$ (cf. also the definitions in the section thickness).*

Lemma 4.2. *Let H be a set satisfying the same condition as the elements of $\mathcal{H}$ in Theorem 4.1. Let n denote the usual normal vector of the boundary (pointing outwards). Then*

$$\int_{\partial H} v_{x,H}(P)dn(P) = -\lambda(H),$$

(where λ means the usual Lebesgue measure on $\mathbf{R}^n$) except for a μ-zero set of $x \in S^{n-1}$ where μ satisfies the condition in Theorem 4.1.

Both for the dimension of the space we work in and for the normal vector of ∂H we will use the notation n. However this will not lead to confusion, because when we talk about the normal vector it will be always denoted as $n(P)$ where P is the point in which the normal vector was considered.

Note that it is easy to prove the Lemma 4.2 even for every x if H is a polyhedron: take the $n-2$-skeleton K_H of H and project it onto ∂H in direction $-x$ and integrate on the parts this projection divides ∂H. It is easy to check that we get the negative of the volumes of strips in direction x on these parts. This way we get a much easier proof of Theorem 3.1.

We will consider a 'surface'-measure on ∂H. Take the coordinating maps $f_1, \ldots, f_N$ of the N surfaces the union of which is ∂H. Now take any Borel subset T of ∂H (the topology is defined by the coordinating maps f_i: G is open in ∂H iff for all i $f_i^{-1}(G)$ is relatively open in U_i), i.e. $T_i := f_i^{-1}(T)$ is a Borel subset for every i. If P is in U_i, let $J_i(P)$ denote the volume of parallelepiped defined by vectors $\partial_1 f_i(P), \ldots, \partial_{n-1} f_i(P)$. Then the measure of T is defined as

$$\sum_{i=1}^{N} \int_{T_i} J_i(P) d\lambda(P)$$

where λ denotes the Lebesgue measure on $\mathbf{R^{n-1}}$. It is easy to check that this is a finite Borel-measure on ∂H.

Proof of Lemma 4.2. First suppose that H is convex and its boundary is a closed C^1-manifold of dimension $n-1$ (from this point smooth will mean C^1). In this case the vector field $v_{x,H}(P)$ is in $C^1(H)$. Calculating the divergence of $v_{x,H}(P)$ we get -1, because this is the same as in the case when x is vertical (because the matrix of the derivatives gets conjugated with the matrix of a rotation that makes x vertical and this does not alter the trace which is the divergence). So the theorem of Gauss-Ostrogradski gives that

$$\int_{\partial H} v_{x,H}(P) dn(P) = \int_{\partial H} v_{x,H}(P) dn(P) = \int_{S} \operatorname{div}(v_{x,H})(P) d\lambda = -\lambda(H)$$

and this is exactly what we wanted to prove.

If H is convex, then the proof is easy again by approximating H with convex sets having a boundary that is a closed smooth-manifold of dimension $n-1$. It can be done by modifying H only in the ε neighbourhood of the boundaries of the embedded manifolds.

Note that till this point the proof worked for every x.

In the general case we can assume again that ∂H is a closed smooth-manifold. Now $v_{x,H}(P)$ is not necessarily continuous. It is easy to see that the set of points where $v_{x,H}(P)$ is not continuous is the subset of the tangent hyper-planes parallel to x. First let's investigate the points of ∂H with such tangent hyper-planes: let's denote their set by A_x.

Take all the directions that are orthogonal to x. This is a $S^{n-2} \subset S^{n-1}$. Then A_x is the inverse image of this S^{n-2} by the Gauss-map $n(P)$. If all the points in the S^{n-2}

are regular values of $n(P)$ (i.e. the derivative-matrices are non-degenerating in the points of A), then A is a sub-manifold of H with co-dimension 1. Take lines parallel to x passing through the points of A. This way we divide H into finitely many parts satisfying the same condition, but also having the property that in the interior of any of these parts H' $v_{x,H'}(P)$ is smooth and it is the same as $v_{x,H}(P)$. Now on the boundaries not being part of the boundary of H the surface-integrals are zeros, because the normal vectors in the points of these new boundaries are orthogonal to x.

Notice that we do not actually need that A is a sub-manifold. It is enough that it is Lebesgue-zero. To see this we take a smooth open B with measure less then ε containing A_x (this is easy because A_x is compact, so B can be chosen to be a finite union of balls and then it is easy to make it smooth). After this we take lines parallel to x at every point of B: the union of these is B' and take $H \backslash B'$. From this point we are in a situation similar to the one described in the previous paragraph. So the formula is true for the closure of $H \backslash B'$ but from its boundary we only need $\partial H - B$. Then we tend with ε to zero: from the surface-area we threw out at most ε and from the volume at most εD where D is the diameter of H, which proves the proposition.

So we only have to prove that A_x is Lebesgue-zero for μ-almost every x. First the set of 'exceptional' vectors v in S^{n-1} for which $n^{-1}(v)$ is not Lebesgue zero is countable. Indeed, otherwise there would more than countably many vectors v for which $n^{-1}(v)$ has measure more than $\frac{1}{n}$ which contradicts that the measure of ∂H is finite. So the vectors corresponding to S^{n-2}-s that pass through these points are in a countable union of S^{n-2}-s corresponding to the 'exceptional' directions. Now we claim that among the remaining vectors there can be at most countably many 'bad' ones. Suppose indirectly that there are more than countably many 'bad' vectors x. Then there exists k for which the measure of A is greater than $\frac{1}{k}$ for more than countably many vectors x. Take countably many of these and consider the corresponding S^{n-2}-s. Throw out from each of them the points that belongs to at least two S^{n-2}-s: this way we left out countably many points from each of them. The measures of the inverse images are still the same, because the inverse images of these vectors in the pairwise intersections had measure 0. But this is impossible, because the measure of ∂H is finite. This finishes the proof. □

A natural question is when this equality holds for all x. If we assume that the points in ∂H for which the derivative-matrix of $n(P)$ is zero has measure zero, then probably a proof analogous to the proof of Saard's Theorem in [4] on pages 16-19 can show that for every x A_x has measure zero.

Now we can prove Theorem 3.1 with a variant of Fubini: we use it when one of the integrals is a surface-integral and the other is a normal integral. It can be applied without any difficulty: both measure μ and the surface-measure on ∂H being σ-finite (in fact finite) and $v_{x,H}(P)n(P)$ being measurable as the function of both x and P.

Proof of Theorem 4.1 Note the union H itself satisfies the condition we required for the elements of $\mathcal{H}$. So let K_H denote more generally then before the union of the boundaries of the smooth parts in ∂H. By assumption this does not count in the surface-integral. H satisfies the conditions of Lemma 4.2, so

$$\lambda(H) = -\int_{S^{n-1}} \int_{\partial H} v_{x,H}(P) dn(P) d\mu(x)$$

(note that x was fix in the Lemma 4.2). Thus by changing the order of integration

$$\lambda(H) = -\int_{\partial H} \int_{S^{n-1}} v_{x,H}(P) d\mu(x) dn(P) = -\int_{\partial H - K_H} v_{P,H,\mu} dn(P)$$

so this is at least $s(H) \cdot \inf\{-v_{P,A,\mu} n(P) : P \in \partial H - K_H\}$. This is at least T_μ because $-v_{P,H,\mu} n(P) = T_{P,H,\mu} \geq T_{P,A,\mu}$ by Remark 2.3, where A denotes the set in the union to which the point P belongs. The proof is finished. □

4.2 The average radius-vector in the uniform case

We investigate further the definitions and theorems of the previous section in particular cases, namely when μ is uniform. The main idea is that the the vector field $v_x(Q)$ may not be smooth, but its average $\int_{S^{n-1}} v_x(Q)\lambda(x)$ (λ denoting the usual normed measure on S^{n-1}, to which we refer as uniform) can be smooth. Let's denote it by $v_{Q,H}$).
In the case of the unit square it is easy to calculate $v_{(x,y),[0,1]^2}$ if $0 < x < 1$, $0 < y < 1$. Let's define F as

$$F(p, q) := \frac{1}{4\pi} q \log\left(1 + \frac{p^2}{q^2}\right) + \frac{1}{2\pi} p \operatorname{arctg}\frac{q}{p}$$

With this notion a calculation similar to the one in the proof of Theorem 5.1 in the next section shows that

$$\left(v_{(x,y),[0,1]^2}\right)_1 = F(1-x, y) + F(1-x, 1-y) - F(x, y) - F(x, 1-y)$$

$$\left(v_{(x,y),[0,1]^2}\right)_2 = F(1-y, x) + F(1-y, 1-x) - F(y, x) - F(y, 1-x)$$

The main point in this is that this is a smooth vector-field inside the square (although what we integrated was not smooth!) and we expect that its divergence is -1 at every point inside because of the original divergences being -1 (where the original vector-field was smooth). The formal calculation:

$$\partial_p F(p, q) = \frac{1}{2\pi} \operatorname{arctg}\frac{q}{p}$$

so we get for the divergence

$$\partial_x \left(v_{(x,y),[0,1]^2}\right)_1 + \partial_y \left(v_{(x,y),[0,1]^2}\right)_2 =$$

$$= -\frac{1}{2\pi}\left(\operatorname{arctg}\frac{y}{1-x} + \operatorname{arctg}\frac{1-y}{1-x} + \operatorname{arctg}\frac{y}{x} + \operatorname{arctg}\frac{1-y}{x}\right) -$$
$$-\frac{1}{2\pi}\left(\operatorname{arctg}\frac{x}{1-y} + \operatorname{arctg}\frac{1-x}{1-y} + \operatorname{arctg}\frac{x}{y} + \operatorname{arctg}\frac{1-x}{y}\right)$$

and this is -1 as can be seen from pairing up $\operatorname{arctg}\frac{p}{q}$ with $\operatorname{arctg}\frac{q}{p}$ their sum being $\frac{\pi}{2}$.
Now the natural question is that is the same true for any 'nice' set in $\mathbf{R}^n$? We prove it in the case of convex polyhedra using ideas and results from the thickness section.

Theorem 4.2. *If H is a convex polyhedron then the average radius-vector is a smooth vector-field (smooth meaning that it has continuous derivatives on H).*

We will use some of the notations of Lemma 2.5. To simplify the proof we introduce a notation.

Definition 4.3. *Let H be a convex polyhedron, F a closure of a face of it, $P \in intH$ and $x \in S^{n-1}$ a direction we get if we connect P as starting point with a point in F. Let $A_{P,F,x}$ denote the following $\mathbf{R}^n \to \mathbf{R}^n$ linear transformation:*
for $h \in \mathbf{R}^n$ take point Q such that $\overrightarrow{PQ} = h$, find the unique points X and Y on the hyper-plane spanned by F such that $\overrightarrow{PX}$ and $\overrightarrow{QY}$ are parallel to x and finally take $A_{P,F,x}(h)$ to be $\overrightarrow{QY} - \overrightarrow{PX}$.

Note that this definition is correct only if H is convex. Indeed, the problems with a concave polyhedron are again the hyper-planes of the faces intersecting the interior of H. Pairing these points with the corresponding faces the projection does not make sense. We also remark that the linearity of $A_{P,F,x}$ can be checked trivially. Now let's continue with a further definition:

Definition 4.4. *For a fix P and F let $a_{P,F}$ denote the supremum of the norms of the linear transformations $A_{P,F,x}$ where x takes all its possible values.*

We remark that the supremum is in fact a maximum because F is closed and the norm of $A_{P,F,x}$ is continuous in x. Let's prove the continuity of $a_{P,F}$:

Lemma 4.5. *$a_{P,F}$ is continuous for $P \in intH$.*

Proof of Lemma 4.5. An easy consideration shows that for a point P the norm of $A_{P,F,x}$ depends only on the angle of x and the hyper-plane spanned by F, the function that can be defined this way is continuous and it is the same for all points in int$H$. We know already that for every ε there is a δ such that if $d(P,Q) < \delta$ then for every $R \in F$ the angle PRQ is less than ε, and this clearly proves the lemma. □

Proof of Theorem 4.2. Take a point P in the interior of H. Let d denote its distance from ∂H. Now take a point Q in the interior and suppose that the distance of P and Q is less than $\frac{d}{2}$. This means that the distance of any point of the line segment PQ from ∂H is greater than $\frac{d}{2}$. Let's denote the distance of P and Q by r and the vector $\overrightarrow{PQ}$ by h. It is easy to see that for any point $X \in \partial H$ the angle PXQ is at most $2\text{arctg}\left(\frac{x}{d}\right)$. This means that if we choose ε to be more than $2\text{arctg}\left(\frac{r}{d}\right)$ and consider $K_H(P)_\varepsilon$, then if we start half-lines from P and Q in any direction not contained in $K_H(P)_\varepsilon$, we will meet the same face of H. (this argument was the repetition of the one used in the proof of Lemma 2.5). We choose ε to be $3\text{arctg}\left(\frac{r}{d}\right)$. Let's consider the difference of the average radius-vectors in points P and Q:

$$v_{P,H} - v_{Q,H} = \int_{S^{n-1}} (v_{x,H}(P) - v_{x,H}(Q)) d\lambda(x) =$$

$$= \int_{K_H(P)_\varepsilon} (v_{x,H}(P) - v_{x,H}(Q)) d\lambda(x) + \int_{S^{n-1} - K_H(P)_\varepsilon} (v_{x,H}(P) - v_{x,H}(Q)) d\lambda(x)$$

Let's estimate first the absolute value of the first addend. This is easy if we note that $a_{Q,F}$ is bounded if Q is in the closed ball of radius $\frac{d}{2}$ and centre P by Lemma **??**. If we take all the finitely many faces of H these bounds corresponding to the faces have a common bound, so if K_H is such a bound for them then $|v_{x,H}(P) - v_{x,H}(Q)| \leq Kr$ if $r \leq \frac{d}{2}$. This yields that the first integral is at most $Kr\lambda\Big(K_H(P)_\varepsilon\Big)$. The second integral can be rewritten as

$$\int_{S^{n-1}-K_H(P)_\varepsilon} (v_{x,H}(P) - v_{x,H}(Q))d\lambda(x) =$$

$$\int_{S^{n-1}-K_H(P)_\varepsilon} A_{P,F_{P,x},x}(h)d\lambda(x) =$$

$$\int_{S^{n-1}} A_{P,F(P,x),x}(h)d\lambda(x) - \int_{K_H(P)_\varepsilon} A_{P,F(P,x),x}(h)d\lambda(x)$$

where $F(P,x)$ denotes the face where the half-line starting from P in direction x meets the polyhedron. The set of x-s for which this is not unique is exactly $K_H(P)$, but this does not count because this set is λ-zero. So finally we only have to estimate $\int_{K_H(P)_\varepsilon} A_{P,F(P,x),x}(h)d\lambda(x)$ but this is basically same as $\int_{K_H(P)_\varepsilon}(vx, H(P) - v_{x,H}(Q))d\lambda(x)$ and it can be estimated the same way. So finally what we got is

$$\Big|v_{P,H} - v_{Q,H} - \Big(\int_{S^{n-1}} A_{P,F(P,x),x} d\lambda(x)\Big)(h)\Big| \leq 2Kr\lambda\Big(K_H(P)_\varepsilon\Big)$$

and this is exactly that $v_{P,H}$ is differentiable at point P because $\lambda(K_H(P)_\varepsilon)$ goes to zero as r goes to zero. Indeed, ε goes to zero if r goes to zero (from the definition of ε) and so what we claimed is true because $\lambda(K_H(P)) = 0$ (this is again the same as in the case Lemma 2.5). It is easy to check that the derivatives are continuous on H. We finished the proof.

□

We remark that from the properties of λ we only used that it is 0 on the spheres of S^{n-1}. Now let's proceed with the divergence.

Proposition 4.6. *If H is convex, then the divergence of the vector-field of the average radius-vectors is -1.*

Proof of Proposition 4.6 In the proof of the previous statement we got that the matrix of the derivative at point P is $\int_{S^{n-1}} A_{P,F(P,x),x} d\lambda(x)$. Now we have to calculate the trace of this. This is simply $\int_{S^{n-1}} \mathrm{Tr}(A_{P,F(P,x),x})d\lambda(x)$. It is enough to prove that $\mathrm{Tr}(A_{P,F(P,x),x}) = -1$. This can be proved the same way as in Lemma 4.2. This finishes the proof. □

If H is not convex then the average radius-vector is not necessarily smooth. As an example take the union of 3 square in a shape of an 'L' and the midpoint P of a side of a square in the interior of the 'L' (cf. figure 3.).
Now if we move to the right with x, then the second coordinate of the average radius-vector is

$$\frac{1}{2}\Big(\mathrm{arctg}\Big(2(1-x)\Big) + \mathrm{arctg}\Big(2(1+x)\Big)\Big) + \frac{1}{2}\Big(\mathrm{arctg}\Big(2(1-x)\Big) + \mathrm{arctg}(2x)\Big) +$$

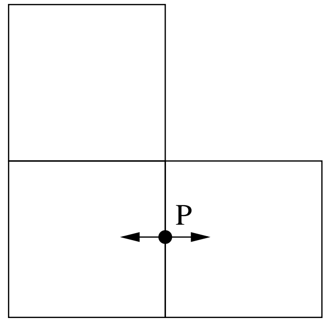

Figure 3:

$$+\frac{3}{2}\Big(\mathrm{arctg}\Big(\frac{2(1+x)}{3}\Big)-\mathrm{arctg}(2x)\Big)+(1-x)\Big(\log\Big(1+\frac{1}{4(1-x)^2}\Big)\Big)+$$

$$+(1+x)\Big(\frac{1}{2}\log\Big(1+\frac{1}{4(1+x)^2}\Big)+\frac{1}{2}\log\Big(1+\frac{9}{4(1+x)^2}\Big)\Big)$$

and if we move to the left with $-x$ where $x<0$ then the second coordinate of the average radius-vector is

$$\frac{1}{2}\Big(\mathrm{arctg}\Big(2(1-x)\Big)+\mathrm{arctg}\Big(2(1+x)\Big)\Big)+\frac{1}{2}\Big(\mathrm{arctg}\Big(2(1-x)\Big)+\mathrm{arctg}(2x)\Big)+$$

$$+\frac{3}{2}\Big(\mathrm{arctg}\Big(\frac{2(1+x)}{3}\Big)-\mathrm{arctg}\Big(\frac{2x}{3}\Big)\Big)+(1-x)\Big(\log\Big(1+\frac{1}{4(1-x)^2}\Big)\Big)+$$

$$+(1+x)\Big(\frac{1}{2}\log\Big(1+\frac{1}{4(1+x)^2}\Big)+\frac{1}{2}\log\Big(1+\frac{9}{4(1+x)^2}\Big)\Big)+$$

$$+(-x)\Big(\frac{1}{2}\log\Big(1+\frac{9}{4x^2}\Big)-\frac{1}{2}\log\Big(1+\frac{1}{4x^2}\Big)\Big)$$

The difference between the derivatives of the two functions is the derivative of their difference, i.e. the derivative of

$$\frac{3}{2}\Big(-\mathrm{arctg}\Big(\frac{2x}{3}\Big)+\mathrm{arctg}(2x)\Big)-x\Big(\frac{1}{2}\log\Big(1+\frac{9}{4x^2}\Big)-\frac{1}{2}\log\Big(1+\frac{1}{4x^2}\Big)\Big)$$

which is $2-\frac{1}{2}\log 9\neq 0$. This shows that the partial derivative in the direction of the x-axis does not exist.

However, it is also possible that the average radius-vector is smooth in the case of a concave set. The easiest example is a ring bounded by two concentric circles. We have a conjecture about when the vector-field of the average radius-vectors will be smooth.

Conjecture 4.7. *If H is a set satisfying the condition in Theorem 4.1, then the average radius-vector is smooth if and only if there is no tangent hyper-plane of ∂H that touches it in at least two points.*

5 Circles and squares

5.1 The result for squares

Theorem 5.1. *If H is a union of unit squares, then the ratio of the perimeter and area of H cannot exceed $\frac{2\pi}{\frac{1}{2}\log 2+\frac{\pi}{4}} \sim 5.551$.*

Proof of Theorem 5.1. For μ we choose the uniform distribution on the interval $[0, 2\pi]$ (we will use the notation $\frac{dx}{2\pi}$ for it). We have to calculate the smallest average thickness with respect to $\frac{dx}{2\pi}$ for all the pairs (P, H) where H is a unit square and P is a point on its boundary. Because μ is uniform, we can suppose that the square has horizontal and vertical sides and that P lies on its horizontal side. If P divides the side on which it lies into two parts of length q and p (p is to the right from P, $p+q=1$) then we can calculate $t_{P,H}(x)$ easily:

$$t_{P,H}(x) = p\mathrm{tg}x \quad (0 \le x \le \mathrm{arctg}\frac{1}{p})$$

$$t_{P,H}(x) = 1 \quad (\mathrm{arctg}\frac{1}{p} \le x \le \pi - \mathrm{arctg}\frac{1}{q})$$

$$t_{P,H}(x) = q\mathrm{tg}(\pi - x) \quad (\pi - \mathrm{arctg}\frac{1}{q} \le x \le \pi)$$

$$t_{P,H}(x) = 0 \quad (\pi \le x < 2\pi)$$

So we can calculate $T_{P,H,\frac{dx}{2\pi}}$:

$$\int_0^{\mathrm{arctg}\frac{1}{p}} p\mathrm{tg}x\frac{dx}{2\pi} + \int_{\mathrm{arctg}\frac{1}{p}}^{\pi-\mathrm{arctg}\frac{1}{q}} \frac{dx}{2\pi} + \int_{\pi-\mathrm{arctg}\frac{1}{q}}^{\pi} q\mathrm{tg}(\pi - x)\frac{dx}{2\pi} =$$

$$= \frac{[-p\log\cos(x)]_0^{\mathrm{arctg}\frac{1}{p}} + \left(\pi - (\mathrm{arctg}\frac{1}{p} + \mathrm{arctg}\frac{1}{q})\right) + [q\log\cos(\pi - x)]_{\pi-\mathrm{arctg}\frac{1}{q}}^{\pi}}{2\pi}$$

Now using the elementary $\cos^2 x = \frac{1}{1+\mathrm{tg}^2 x}$ what we get finally is

$$\frac{p\frac{1}{2}\log(1+\frac{1}{p^2}) + q\frac{1}{2}\log(1+\frac{1}{q^2}) + \left(\pi - (\mathrm{arctg}\frac{1}{p} + \mathrm{artan}\frac{1}{q})\right)}{2\pi}$$

Note that the formula makes sense for $p=0$ (and $q=0$) because

$$\lim_{p\to 0+} p\log(1+\frac{1}{p^2}) = \lim_{p\to 0+}\log(1+\frac{1}{p^2})^p = \log 1 = 0$$

and

$$\lim_{p\to 0+}\mathrm{arctg}\frac{1}{p} = \frac{\pi}{2}$$

We have to find the minimum of this formula with condition $0 \le p, q \le 1$ and $p+q=1$. If we prove that the function $p\frac{1}{2}\log(1+\frac{1}{p^2}) - \mathrm{arctg}\frac{1}{p}$ is concave then the minimum will be at $p=0$, $q=1$ or $p=1$, $q=0$. But that can be easily verified, because its derivative is $\frac{1}{2}\big(\log(1+\frac{1}{p^2}) - \frac{1}{p^2+1}\big) + \frac{1}{p^2+1} = \log(1+\frac{1}{p^2})$ and it is decreasing because $1+\frac{1}{p^2}$ is decreasing for $0 \le p$.

This means that in this case $\frac{1}{T_{\frac{dx}{2\pi}}} = \frac{2\pi}{\frac{1}{2}\log 2 + \frac{\pi}{4}}$ which is slightly less then 5.551. □

5.2 Restricted case of squares

Theorem 5.2. *If H is a union of unit squares the sides of which are parallel or have degree $\frac{\pi}{4}$, then the ratio of the perimeter and area of H cannot exceed 4.*

Proof of Theorem 5.2. Let the axis x be parallel with one of the sides of the squares. Take strips at directions 0, $\frac{\pi}{2}$, π and $\frac{\pi}{2}+\pi$ and let all the weights be $\frac{1}{4}$. Now we claim that for any fixed point P on the boundary of the square H

$$\sum_{i=1}^{4} \frac{1}{4} t_{P,H}(\alpha_i) = \frac{1}{4}$$

so with this choice of measure μ $T_{P,H,\mu} \equiv \frac{1}{4}$ and thus $T_\mu = \frac{1}{4}$ which proves the theorem. □

There are two possible cases. The first case is when there are two nonzero summand in the average thickness belonging to the line segments starting from P. In this case both directions with non-zero thickness have angle $\frac{\pi}{4}$ with the side containing P. If P divides the side into line segments of length x and $1-x$, then the lengths of the longest segments in the square in these two directions are $\sqrt{2}x$ and $\sqrt{2}(1-x)$, so the average thickness is $\frac{1}{4}(\sqrt{2}\sin\frac{\pi}{4}x + \sqrt{2}(1-x)\sin\frac{\pi}{4}) = \frac{1}{4}$ as we claimed.

The second case is when there is only one nonzero summand in the average thickness. In this case the direction with non-zero thickness is orthogonal to the side containing P. So the average thickness is $\frac{1}{4}$ in this case too. We finished the proof. □

A similar result for the regular n-gons can be found in the section of extremal measures.

5.3 The case of circles

So far circle is the only set in the plane for which we know the exact supremum of the perimeter-area ratio for a finite union of them. Note that this is easy from Theorem 3.1 and Proposition 6.4: by taking $\frac{dx}{2\pi}$ the average thickness is the same in every point of the perimeter of the circle. However we give here a nice elementary proof for the case of circles. The idea of the proof is somewhat similar too the main idea of a much more complicated proof in [2].

Theorem 5.3. *If H is a finite union of circles with diameter 1, then the ratio of the perimeter and area of H cannot exceed 4.*

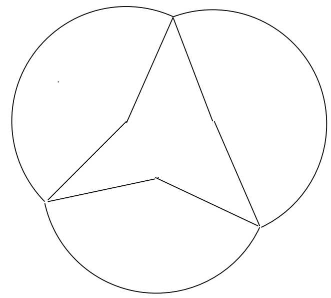

Figure 4:

Proof of Theorem 5.3 Take an arc a in ∂H and the centre O_a of the circle corresponding to the arc a. Take the slice of the circle defined by a and O_a (i.e. the radiuses of the circle connecting O_a with the points of a). This can be seen on figure 4. The area of the slice is exactly $\frac{l(a)}{4}$ where $l(a)$ denotes the length of arc a. So we finish our proof if we can show that the interior of these slices of circles are disjoint for different a-s. Suppose that it is not the case: then we can find different arcs a and b, centres O_a and O_b and points A and B on the arcs (not being endpoints) such that O_aA intersects O_bB in a point different from O_a and O_b. This means that ABO_aO_b is a convex (non-degenerate) quadrangle. From the triangular inequality it is easy to derive that $1 = O_aA + O_bB > O_aB + O_bA$. This means that either O_aB or O_bA is less then $\frac{1}{2}$, contradicting the fact that both of them are in ∂H. This finishes the proof. □

Finally we show a funny connection between the case of squares and circles. This was discovered by Dömötör Pálvölgyi.

Proposition 5.1. *If a the perimeter-area ratio of the finite union of unit squares cannot be more than 4, then the same is true for the circles with diameter 1.*

Proof of Theorem 5.1. Take a finite union of circles with diameter 1, denote it by H. Now replace every circle with a union of squares with diameter 1 and having the same centre as the circle and denote the new union we get this way by H'. If the area of the union of the squares replacing a fixed circle is more than $\frac{\pi}{4} - \varepsilon$, then the union of these squares have perimeter more than $\sqrt{2}(\pi - 4\varepsilon)$. This is true because the ratio of the perimeter and area of a union of squares of diameter 1 with the same centre is $4\sqrt{2}$ similarly to Proposition 8.2. This yields that the perimeter of the union of the squares with which we replaced the circles is more than $\sqrt{2}$ times the perimeter of the union of circles minus $4\sqrt{2}N\varepsilon$ where N is the number of circles in the union. So finally using the inequality we supposed to be true in the case of squares of diameter 1 we get that

$$\frac{\sqrt{2}p(H) - 4\sqrt{2}N\varepsilon}{a(H)} \leq \frac{p(H')}{a(H')} \leq 4\sqrt{2}$$

which gives the desired result if ε goes to 0. □

6 Extremal measures

6.1 The definition and the case related to the original problem

Naturally arises the question if $\frac{dx}{2\pi}$ is the best measure for the unit squares. The answer is yes and the proof is based on an idea of Dömötör Pálvölgyi. We start with a definition.

Definition 6.1. *Let $\mathcal{H}$ be a set of subsets of $\mathbf{R}^n$. We say that λ is an* ***extremal measure of*** *$\mathcal{H}$ if*

$$\inf\{T_{H,\lambda} : H \in \mathcal{H}\} = \sup\{\inf\{T_{H,\mu} : H \in \mathcal{H}\} : \mu \in \boldsymbol{P}(S^{n-1})\}$$

Theorem 6.1. *For any μ probability Borel measure on $[0, 2\pi)$ and ε there is a pair (P, H) where H is a unit square and P is a point on its perimeter such that*

$$T_{P,H,\mu} \leq \frac{\frac{1}{2}\log 2 + \frac{\pi}{4}}{2\pi} + \varepsilon$$

.

Proof of Theorem 6.1. First we define the average thickness in the vertices of the square too. The definition makes sense if we choose a 'tangent-line' at the vertices, too. We choose them to be one of the sides. It is easy to see that with this choice the average thickness is continuous for any μ probability measure. So it is enough to prove the theorem for this extended average thickness. Take P to be a fixed point and consider all the squares having P among its vertices. Take the positive orientation of the boundary of a square and associate with it its side-vector starting from P. This way we parametrised these squares with the unit-vectors in the plane or using other words we parametrised the squares with the interval $[0, 2\pi)$. Let's denote the square belonging to $\alpha \in [0, 2\pi)$ by H_α. Now we claim that

$$\int_0^{2\pi} T_{P,H_\alpha,\mu} \frac{d\alpha}{2\pi} = \frac{\frac{1}{2}\log 2 + \frac{\pi}{4}}{2\pi}$$

this will certainly prove our theorem. The integral can be rewritten using the definition of the average thickness as the double integral

$$\int_{[0,2\pi)} \int_{[0,2\pi)} t_{P,H_\alpha}(\beta) d\mu(\beta) \frac{d\alpha}{2\pi}$$

and by the theorem of Fubini this is the same as

$$\int_{[0,2\pi)} \int_{[0,2\pi)} t_{P,H_\alpha}(\beta) \frac{d\alpha}{2\pi} d\mu(\beta)$$

Now the internal integral $int_{[0,2\pi)} t_{P,H_\alpha}(\beta) \frac{d\alpha}{2\pi}$ can be rewritten using the definition of H_α as

$$\int_{[0,2\pi)} t_{P,H_0}(\beta - \alpha) \frac{d\alpha}{2\pi} = \int_{[0,2\pi)} t_{P,H_0}(\gamma) \frac{d\gamma}{2\pi}$$

where in the last step we replace our variable α by introducing a new variable $\gamma := \beta - \alpha$. Now this last integral (as we calculated earlier) is $\frac{\frac{1}{2}\log 2 + \frac{\pi}{4}}{2\pi}$ which proves our equality and also the theorem. □

What we proved can be rephrased by using our definition:

Corollary 6.2. *If $\mathcal{H}$ consists of all the unit squares in the plain, then $\frac{dx}{2\pi}$ is an extremal measure of $\mathcal{H}$.*

Remark 6.3. *The proof also works in a more general case: namely, if $\mathcal{H}$ is closed with respect to the rotations (that is, if σ is a rotation and $H \in \mathcal{H}$ then $\sigma H \in \mathcal{H}$), then the extremal measure is the usual Lebesgue measure on S^{n-1}.*

6.2 The case when $\mathcal{H}$ contains only the translated copies of a fixed set

In this section we consider the case when $\mathcal{H}$ has only the translated copies of a fixed polyhedron. This means we want to calculate the slimness of a given polyhedron. First we prove a general statement.

Proposition 6.4. *If A is a polyhedron and μ is a Borel probability measure for which $T_{P,A,\mu}$ is constant for all the points P inside the faces, then it is an extremal measure for A.*

Proof of Proposition 6.4. In the proof of theorem 3.1 in section 4 we proved that if we integrate $T_{P,A,\mu}$ on the faces of the polyhedron A and add them up we get the volume of A (and it would be easy to prove it with a slight modification of the proof of Theorem 3.1 in section 3). So if $T_{P,A,\mu}$ is a constant then it must be the ratio of the volume and surface-area of the polyhedron. The same equality shows that for every μ' there is always a P such that $T_{P,A,\mu'}$ is at most this ratio, which finishes our proof. □

6.3 Sphere-like sets

Definition 6.5. *Let A be a set in $\mathbf{R}^n$ that is a closure of an open set and have a boundary consisting of finitely many compact smooth manifolds embedded in $\mathbf{R}^n$. We call A sphere-like, if the uniform distribution on S^{n-1} is an extremal measure for it.*

Conjecture 6.6. *Only the sphere is sphere-like.*

Definition 6.7. *We call a polyhedron A **weakly sphere-like**, if there is a measure μ with the property of proposition 6.4.*

From the proof of Proposition 6.4 it is clear that if A is weakly sphere-like and μ is an extremal measure for A then $\int_{S^{n-1}} t_{P,A,x} d\mu(x) = \frac{s(A)}{v(A)}$ for all P on the boundary of A (and not in the $n-2$ skeleton) where $s(A)$ and $v(A)$ denotes the surface-area and volume of A respectively.

A natural question if every polyhedron is weakly sphere-like. We give an example that this is not the case. First we prove an easy statement:

Proposition 6.8. *If H is a union of translated copies of a sphere-like polyhedron, then the ratio of the surface-area and volume of H cannot exceed the same quantity of the polyhedron.*

Proof of Proposition 6.8. This is a trivial consequence of Theorem 3.1. □

So to find a polyhedron that is not sphere-like it is enough to find one for which the previous statement is not true. We give an example in the plane.

Take a square with side length n, where n is odd, and put on one of its sides $\frac{n+1}{2}$ equilateral triangle with side length 1 starting from the endpoint of the side and having gaps of length 1 between the triangles (figure 5. shows the case of $n = 5$).

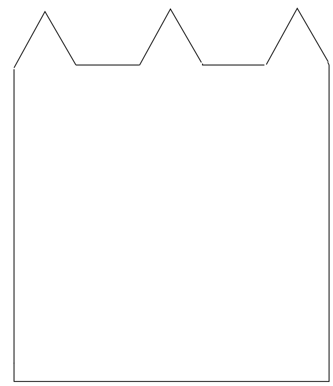

Figure 5:

The area of this set is $n^2 + \frac{n+1}{2}\frac{\sqrt{3}}{4}$ and its perimeter is $3n + 2\frac{n+1}{2} + \frac{n-1}{2}$. Now translate this set with a vector of length 1 and parallel to the side of the square on which the triangles were placed. The area of the union is $n(n+1) + (n+1)\frac{\sqrt{3}}{4}$: that is a growth of $n + \frac{n+1}{2}\frac{\sqrt{3}}{4}$. The perimeter of the new set is $3n + 1 + 2(n+1)$, which means it increased by $\frac{1}{2}n + \frac{5}{2}$. The ratio of the growths is $\frac{\frac{1}{2}n+\frac{5}{2}}{n+\frac{n+1}{2}\frac{\sqrt{3}}{4}}$ which goes to $\frac{\frac{1}{2}}{1+\frac{\sqrt{3}}{8}}$ while the original ratio goes to zero as n goes to infinity. So if n is large enough, the ratio of the growths is bigger, which means that the ratio of the perimeter and area of the union is greater than the original one, which proves that this polygon is not sphere-like.

We also prove a positive statement.

Proposition 6.9. *Triangles, parallelograms and regular polygons are all sphere-like.*

Proof of Proposition 6.9. Let's proceed one by one.

The case of triangles: take a triangle ABC, we denote the side-lengths and the heights by a, b, c and h_a, h_b and h_c as usual. The support of the measure will consist of directions $\overrightarrow{AB}$, $\overrightarrow{BA}$, $\overrightarrow{BC}$, $\overrightarrow{CB}$, $\overrightarrow{CA}$ and $\overrightarrow{AC}$. Let's choose the weights to be $\frac{b}{2(a+b+c)}$, $\frac{a}{2(a+b+c)}$, $\frac{c}{2(a+b+c)}$, $\frac{b}{2(a+b+c)}$, $\frac{a}{2(a+b+c)}$ and $\frac{c}{2(a+b+c)}$. With this choice of μ the average direction is constant: for example, if P is on side AB dividing it into line segments of length x and $1-x$, then the sum of the values of the thickness function at this point are zeros in 4 of the 6 directions, and it is xh_c in direction $\overrightarrow{BC}$ and $(1-x)h_c$ in direction $\overrightarrow{AC}$, so the average thickness is $\frac{c}{2(a+b+c)}(xh_c + (1-x)h_c) = \frac{T_{ABC}}{a+b+c}$ not depending on the choice of P.

The case of parallelograms: take a parallelogram $ABCD$ and denote its side-lengths by p (of side AB) and q (of side BC). Now let's choose the support of μ to be the directions $\overrightarrow{AB}$, $\overrightarrow{BC}$, $\overrightarrow{CD}$ and $\overrightarrow{DA}$. The weights are $\frac{p}{2(p+q)}$, $\frac{q}{2(p+q)}$, $\frac{p}{2(p+q)}$ and $\frac{q}{2(p+q)}$ respectively. Now a calculation even easier then in the previous case shows that the average thickness is constant.

The case of regular polygons: Take a regular n-gon $A_0 \dots A_{n-1}$ such that its vertices are oriented positively in this order and side A_1A_2 has the same direction as the x-axis. Choose the support of μ to be the set of angles $\{\frac{2\pi i}{2n} : 0 \geq i < 2n\}$ and choose all the weights to be $\frac{1}{2n}$. Now for an arbitrary point P on the (lower if there are two of them) horizontal side the thickness function is nonzero in points the points $\frac{2\pi i}{2n}$ where $0 < i < n$. Now because the angle $A_1A_0A_j$ is $\frac{(j-1)\pi}{n}$, it is easy to see that the longest line segment in direction $\frac{2\pi i}{2n}$ has its endpoint on side A_iA_{i+1} (we take A_n to be A_0), so by pairing up the values of the thickness function belonging to $\frac{2\pi i}{2n}$ and $\frac{2\pi(n-i)}{2n}$ we get that the $2n$ times the average thickness is the sum of the heights corresponding to the side containing A_0A_1 of the triangles defined by lines A_0A_1, A_iA_{i+1} and $A_{n-i}A_{n-i+1}$ (using the same argument that was used in the case of triangles). This finishes the proof. □

Corollary 6.10. *If H is a union of translated copies of a triangle/parallelogram/regular polygon, then the ratio of the perimeter and area of H cannot exceed the same quantity of the triangle/parallelogram/regular polygon respectively.*

In the case of regular polygons the proof also works if we take not only the original n-gon, but also if we rotate it with angle $\frac{2\pi}{2n}$. In this case $\mathcal{H}$ contains the translated copies of a regular n-gon and its rotated copy with angle $\frac{2\pi}{2n}$. Applying theorem 3.1 in this case we get a result similar to theorem 5.2.

Corollary 6.11. *If H is a union of translated copies of a regular n-gon and its rotated copy with angle $\frac{2\pi}{2n}$, then the perimeter-area ratio of H cannot exceed the same quantity of the regular n-gon.*

We conclude this section with a natural conjecture:

Conjecture 6.12. *Every convex polyhedron is weakly sphere-like.*

6.4 An equivalent definition for a convex polyhedron being weakly sphere-like

We give an equivalent definition of a polyhedron being spherical. However, so far this have not helped in solving the conjecture of the previous section.

Proposition 6.13. *A convex polyhedron H is spherical if and only if there exist no points $P_1, \dots, P_n \in \partial H$ and numbers $a_1, \dots, a_n$ such that $\sum a_i > 0$ and $\sum a_i t_{P_i,H}(x) \leq 0$ for all x.*

Proof of Theorem 6.13. First we prove that if polyhedron H is weakly sphere-like, then such P_i-s and a_i-s cannot exist. Suppose indirectly that there exist such P_i-s and a_i-s. Take a measure μ proving that H is spherical. Now integrating $\sum a_i t_{P_i,H}(x)$ by μ we get that

$$0 \geq \int_{S^{n-1}} \sum a_i t_{P_i,H}(x) d\mu(x) = \sum a_i \int_{S^{n-1}} t_{P_i,H}(x) d\mu(x) =$$

$$= \sum a_i \frac{s(H)}{v(H)} = \frac{s(H)}{v(H)} \sum a_i > 0$$

which is a contradiction.

To prove the other direction fix points $P_1, \dots, P_n$ in the interior of the faces of H. Consider the vectors $\Big(t_{P_1,H}(x), \dots, t_{P_n,H}(x)\Big)$ for all x. Now consider the closure of the cone generated by these vectors. If this does not contain the vector $(1, \dots, 1)$ then there is a hyper-plane containing the origin and separating the cone and $(1, \dots, 1)$ by the Farkas-lemma (c.f. [5]), but this contradicts our assumption. So this cone must contain this vector.

Take a countable dense set in ∂H: $\{P_1, P_2, \dots\}$. By the previous paragraph for all n there exist directions $x_{n1}, \dots, x_{nk_n}$ and non-negative coefficients $w_{n1}, \dots, w_{nk_n}$ such that $|\sum_{j=1}^{k_n} w_{nj} t_{P_i,H}(x_{nj}) - 1| < \frac{1}{n}$. Suppose that n is such big that every face of H contains a point P_i: we can also suppose without the loss of generality that these are $P_1, \dots, P_f$ (f is the number of faces of H). In this case the sum of the numbers a_j is bounded. Indeed, there exist $a > 0$ such that for all $x \in S^{n-1}$ exists a P_i $(1 \le i \le f)$ for which $t_{P_i,H}(x) > a$. To prove this denote the minimum of the angles of adjacent faces by α and fix a direction $x \in S^{n-1}$. There is a face such that if we place the vector $x \in S^{n-1}$ on that face it points in the direction of the interior of H. Now if the angle of x with this face is less then $\frac{\alpha}{2}$, then it is easy to see that its angle with one of the adjacent faces is more than $\frac{\alpha}{2}$, and this proves certainly our claim. And in this case $\sum w_j < \frac{2}{a}$. So if we define μ_n as w_i in directions x_i and zero otherwise, this sequence of Borel-measures contains a convergent subsequence $\mu_{i_1}, \mu_{i_2}, \dots$ in the weak topology. We claim that its limit, μ proves that H is sphere-like. We remark that obviously $\mu(S^{n-1}) < \frac{2}{a}$ (we get this by applying the definition of the weak convergence in the case when we take the constant 1 function on S^{n-1}).

First calculate the average thickness with respect to μ in points P_i: because $t_{P_i,H}(x)$ is continuous, $\int_{S^{n-1}} t_{P_i,H}(x) d\mu_{i_j} \to \int_{S^{n-1}} t_{P_i,H}(x) d\mu$, but this limit is clearly 1 by the definition of μ_n. Now if P is a fixed point in the interior of a face, then it is easy to see that there exists a constant K (depending on P) such that for all $x \in S^{n-1}$ it is true that $|t_{P,H}(x) - t_{Q,H}(x)| < K d(P,Q)$ (cf. the proof of Theorem 4.2). So trivially estimating

$$\Big| \int_{S^{n-1}} t_{P,H}(x) d\mu(x) - \int_{S^{n-1}} t_{P_i,H}(x) d\mu(x) \Big| <$$

$$\int_{S^{n-1}} |t_{P,H}(x) - t_{P_i,H}(x)| d\mu(x) < \frac{2}{a} K d(P_i, P)$$

so obviously $\int_{S^{n-1}} t_{P,H}(x) d\mu(x) = 1$ and this finishes the proof (by norming μ to 1). □

7 Another result related to the original problem

We will prove the following statement with a different technique:

Theorem 7.1. *If H is a union of unit squares such that every point of H can belong to the interior of at most two squares then the ratio of the perimeter and area of H cannot exceed 4.*

To prove this we will prove the following lemma:

Proposition 7.1. *If T is a polygon with the property that the distance of any of its vertices and of its sides is at most 1, then the ratio of its perimeter and area is at least 4.*

We remark that the condition about T is not the same as having diameter at most 1. The consequence of this lemma is a dual to the original problem.

Corollary 7.2. *If T is the intersection of unit squares, then the ratio of its perimeter and area is at least 4.*

Proof of Corollary 7.2. Indeed, if we take only finitely many squares, then this is obvious, because T is a polygon satisfying the condition in Proposition 7.1.

If we take the the intersection of countably many unit squares, then the perimeter and area of the intersection is the limit of the finite intersections, because all these sets are convex. (Note that the same does not hold for the union instead of intersection.)

Finally if we take an arbitrary intersection, either this has dimension less than n when the area is 0 and we are ready, or it can be obtained by a countable intersection. Indeed, otherwise there is a longer than countable sequence of closed convex sets of dimension n $\{A_\phi\}$ such that $A_\alpha \subset A_\beta$ and $A_\alpha - A_\beta \neq \emptyset$ if $\alpha < \beta$. Take a countable dense set $\{P_1, P_2, \dots\}$ in $\mathbf{R}^n$, then for every ϕ there is an i such that $P_i \in A_{\phi+1} - A_\phi$ because the A_ϕ-s are closures of open sets, which means that $A_\alpha - A_\beta \neq \emptyset \Longrightarrow \text{int}(A_\alpha) - A_\beta \neq \emptyset$. For different ϕ-s the i-s must be different, which contradicts the fact that the sequence was longer than countable. □

Proof of Proposition 7.1. We start with the proof of a lemma.

Lemma 7.3. *For given lines $e_1, \dots, e_n$ and positive real numbers $a_1, \dots, a_n$ let $R(e_1, \dots, e_n, a_1, a_2, \dots, a_n)$ denote the set of convex polygons, the sides of which are parallel to the lines $e_1, \dots, e_n$, the sum of the lengths of the sides parallel to e_i is a_i (it is possible that there is only one such side, then it is simply the length of the side). In this set the value of the area cannot be greater than in the polygon that has two parallel sides to e_i, each having length $\frac{a_i}{2}$ (for all $1 \leq i \leq n$).*

Proof of Lemma 7.3. To prove the lemma we first note that the set of polygons $R(e_1, \dots, e_n, a_1, \dots, a_n)$ is compact as the subset of R^{2n}. To understand this first we have to show how comes $R(e_1, \dots, e_n, a_1, \dots, a_n)$ to be the subset of R^{2n}. It is easy: to a polygon in this set corresponds $2n$ real numbers: basically these are the lengths of the sides, but we have to be careful, because it is possible that the number of sides of a polygon in this set is less than $2n$ (note that it cannot be more than $2n$ because of convexity). To overcome this difficulty we consider the polygon with positive orientation and the sides of it as vectors. Let's take unit vectors $v_1, \dots, v_n$ parallel to $e_1, \dots, e_n$ respectively (we also choose the vectors such that their angles $\phi_1, \dots, \phi_n$ satisfies $0 \leq \phi_1 < \dots < \phi_n < \pi$). So every side of the polygon is of form $x_i v_i$ or $-y_i v_i$ where x_i and y_i are positive numbers. If one of these does not occur (it is not possible that both of them are missing), then we take the coefficient in the missing one to be 0. In this way we get a vector $(x_1, \dots, x_n, y_1, \dots, y_n)$ in R^{2n}. It is clear that this set is bounded (because none of the coordinates can be greater than $\max\{a_1, \dots, a_n\}$ and all of them are nonnegative). It is also closed, because to be in the set $R(e_1, \dots, e_n, a_1, \dots, a_n)$ means that $x_i + y_i = a_i$ and $\sum (x_i - y_i) e_i = 0$, both of these remaining true when taking a limit of a sequence of vectors in R^{2n} satisfying these conditions. This shows that this set is compact.

Now let's take any element $(x_1, \ldots, x_n, y_1, \ldots, y_n)$ in this set and suppose that $x_1 < y_1$ (we can suppose this without the loss of generality). Now we want to move the two sides parallel to e_1 in a direction such that x_1 increases and y_i decreases. It is easy if $y_n > 0$ and $x_2 > 0$, because in this case we take v_1 rotated with $+\frac{\pi}{2}$, let's denote it by n_1 and translate the sides corresponding to x_1 and y_1 with tn_1: this is possible if t is small enough. Now the resulting area is greater then the original one for a small t, because the derivative of the area with respect to t is $y_1 - x_1 > 0$.

A bit different case is when there are zeros among the coordinates. Suppose that $x_1 = 0$ (note that in this case $x_1 < y_1$!) and $y_n \neq 0$. This is possible, because if there is a zero side then there is also a zero side with one of its neighbours being nonzero (because there are at least n-nonzero sides). Now $x_1 = x_2 = \cdots = x_n = 0$ is not possible because then $0 = \sum x_i v_1 = \sum y_i v_i$ but $\sum y_i v_i = 0$ with positive coefficients is impossible: all the vectors v_i are in an open half-plane so any linear combination of them with positive coefficients are in that half-plane, too. So suppose that $x_1 = \cdots = x_k = 0$, but y_n and x_{k+1} are positive. Now take vectors tv_n sv_{k+1} and translate the point belonging to x_1 (it is the same which belongs to $x_2, \ldots, x_k$) both with tv_n and sv_{k+1}. After this translate with sv_{k+1} the endpoint of the side-vector corresponding to y_{k+1} and translate the starting point of it (which can coincide with the end-point) with tv_n. All the other vertices belonging to the sides corresponding to $y_1, \ldots, y_n$ we translate with sv_{k+1}. We also choose t and s such that $tv_n - sv_{k+1}$ is parallel to e_1. Denote the length of $tv_n - sv_{k+1}$ by d. This way we get a new polygon (cf. figure 6.).

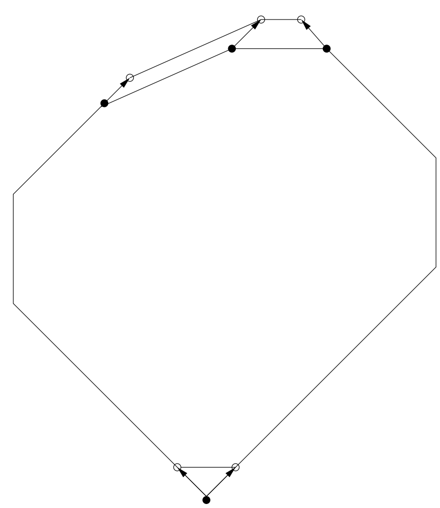

Figure 6:

What is happening now is that we get from the 'polygon' $(0, \ldots, 0, x_{k+1}, \ldots, y_n)$ the new 'polygon' $(d, 0, \ldots, 0, x_{k+1} - s, \ldots, x_n + t, y_1 - d, \ldots, y_{k+1} + s, \ldots, y_n - t)$ and we can also calculate the changing of the area: it decreases with he triangle defined by the two vectors tv_1 and sv_{k+1} the area of which is $O(t^2)$ but increases with more then ct where $c > 0$. To see this it is enough to remark that the new polygon will contain a trapezoid on side $-y_1v_1$ with side-vectors tv_n and sv_{k+1}, so its area is ct with a certain c. This shows that for enough small t the area of the polygon will increase. We proved our proposition. □

Now let's return to the original statement. To prove that we remark first that every polygon in $R(e_1, \ldots, e_n, a_1, a_2, \ldots, a_n)$ has the same distance d_i between the sides parallel to e_i (we keep on considering the polygons as having sides with length zero as described in the proof of the lemma).

There is a funny indirect proof to this fact. Let's take the subset of $R(e_1, \ldots, e_n, a_1, a_2, \ldots, a_n)$ taking only the polygons which sides parallel to e_i have distance d_i. It is a closed subset, so it is compact. So there is a polygon in this subset with maximal area, but the proof above remains valid here. That means that the opposite sides of this polygon are equal, but this polygon is unique which proves what we claimed, because this means there can be only one non-empty subset of the type we considered (the same argument also gives that $R(e_1, \ldots, e_n, a_1, a_2, \ldots, a_n)$ is connected).

Of course there is an elementary direct proof. The distance of the sides parallel to e_1 is $d_1 = n_1 \sum x_i v_i)$ where ab means the scalar product of vectors a and b, and n_1 is the rotation of e_1 with angle $+\frac{\pi}{2}$. Note that x_1v_1 is not needed in the sum but we can leave it there because $n_1v_1 = 0$. We know that $\sum x_iv_i = \sum y_iv_i$ and so $d_1 = n \sum y_iv_i$. By adding the two equalities up we get

$$2d_1 = n_1 \sum (x_i + y_i)v_i = n_1 \sum a_iv_i$$

and this is clearly not depending on the choice of the numbers $x_1, \ldots, y_n$ what is exactly what we wanted to prove.

Now we are ready to prove Proposition 7.1. Take a polygon T satisfying the property in the proposition. Naturally it is in some $R(e_1, \ldots, e_n, a_1, \ldots, a_n)$ (this is obviously true for all convex polygons). Now the area of it cannot be greater than the one in this set having equal opposite sides. By our previous argument this polygon is also satisfying the condition in the statement and we also know that its ratio of the perimeter and area cannot be greater than the same ratio in T (cf. the remark after the lemma). So it is enough to prove that the new ratio is at least 4.

First it is clear that the new polygon T' has a symmetry centre O. Indeed, if its vertices are $P_1, \ldots, P_{2n}$ in this order then P_iP_{i+n} and $P_{i+1}P_{i+n+1}$ $(1 \le i \le n-1)$ have the same midpoint. This is clear because the sides P_iP_{i+1} and $P_{i+n}P_{i+n+1}$ are parallel and have the same length, so the quadrangle $P_iP_{i+1}P_{i+n}P_{i+n+1}$ is a parallelogram. This shows (by induction) that all the diagonals P_iP_{i+n} has the same midpoint O, but then it is clear that it is the symmetry centre. The area of T' is the sum of the areas of the triangles OP_iP_{i+1} $(1 \le i \le 2n$ where $P_{2n+1} = P_1$ by definition). Now the height of this triangle belonging to the side P_iP_{i+1} is the half of the distance between the sides P_iP_{i+1} and $P_{i+n}P_{i+n+1}$ by symmetry. This distance is at most 1 by assumption, so in every triangle the height is at most $\frac{1}{2}$. So the area of T' is at most

$$\sum_{i=1}^{2n} \frac{|P_iP_{i+1}|\frac{1}{2}}{2} = \frac{p(T')}{4}$$

which finishes the proof of our proposition. □

Proof of Theorem 7.2. Now using Proposition 7.1 let's prove the theorem in this section. Let H be a union of a finite number of squares such that every point belongs to the interior of at most 2 squares. Let's consider all the pairs of these squares and their intersections. These are polygons with disjoint interiors, all of which are satisfying the condition in our proposition. Now if there are N squares, the area of H is N minus the areas of the polygons. The perimeter of H is $4N$ minus the perimeter-s of the polygons, because a side of such a polygon is either not on the boundary of H so we have to subtract it or it is, but then it belongs to two squares although it occurs in the perimeter only once, which again means that we have to subtract it exactly once. So if we have to subtract a from N we have to subtract at least $4a$ from $4N$ by using the proposition for all the polygons but this just shows that the ratio cannot be greater than $\frac{4N-4a}{N-a} = 4$. That was our statement. □

8 Examples and counterexamples

During the work many interesting conjectures emerged. Interesting examples and counterexamples were found in many cases, here give some of them. First we take problems that contain the original problem as a special case.

Proposition 8.1. *There exists a convex set in the plane for which it is not true, that any finite union of congruent copies of it have perimeter-area ratio less or equal than the original one.*

We remark that to give a concave example is very easy. We even have an example of a concave 'nice set' when there exists no bound for the perimeter-area ratio of the finite union of congruent copies of he set. It is a 'hyperbolical triangle with angles 0' (see figure 7.). Let the distance between its centre and vertices be 1. Take the union

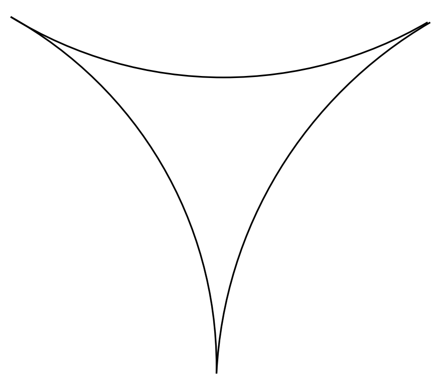

Figure 7:

of n copies of it rotated with angles $\frac{2\pi i}{n}$ $(0 \le i < n)$ around its symmetry centre. It is easy to verify that the area of this union goes to π while the perimeter goes to infinity when n goes to infinity.

Proof. Take a unit square and cut off exactly one of its vertices with line e: we get a pentagon this way. Note that if the line is at distance more than $\frac{1}{2}$ from the centre of the square, then the perimeter-area ratio of the pentagon is less than 4 (cf. figure 8.). Indeed, if we divide the pentagon into five triangles from the centre of the square,

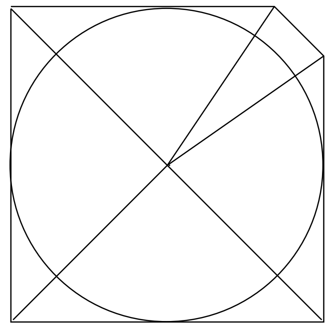

Figure 8:

then the areas of the triangles are $\frac{a}{4}$ where a denotes the side of the triangles that also the side of the pentagon. This is true because the height belonging to these side is the distance of O and an original side of the square that is $\frac{1}{2}$. The fifth has area more than $\frac{a}{4}$ (with the same notation) because the height belonging to this side is more than $\frac{1}{2}$ by our assumption. This shows that the area of the pentagon is more than the quarter of its perimeter. This is exactly what we claimed.

Now the union of the pentagon and its reflection to the centre of the original square is the original unit square, thus the ratio of the perimeter and area is more than it was originally (now being 4). □

The main idea in the proof was that there exits convex sets A and B such that $A \subset B$ but the perimeter-area ratio of B is the greater than of A. We remark that if A or B is a circle, then it is true that the perimeter-area ratio is greater in the case of A. The proof is left to the reader.

Another natural stronger (and more naive) conjecture was that if H is a union of unit squares, then its perimeter-area ratio is strictly less then 4 unless it is union of 'almost-disjoint' squares which means that any two can have at most one point in common. This proved to be false again because the following is true:

Proposition 8.2. *If H is a finite union of unit squares which all have the same centres, then its perimeter-area ratio is exactly 4.*

Proof. Let O denote the common centre of the squares and let $A_0 \dots A_{4n-1}$ be the vertices of polygon H. Now the area of this polygon is the sum of the areas of triangles OA_iA_{i+1} ($0 \leq i < 4n$ and we take $4n$ to be 0). But this area is exactly $\frac{A_iA_{i+1}}{4}$ because the distance of the side A_iA_{i+1} and O is $\frac{1}{2}$. This proves the proposition. □

After this we can easily modify our conjecture about when will equality hold in the original conjecture. To make it shorter we use the word star for a finite union of unit squares with the same centre.

Conjecture 8.3. *A finite union of unit squares have perimeter-area ration less then 4 unless it is an almost-disjoint union of stars, i.e. the union of squares with the same centre.*

We can phrase and prove Proposition 8.2 in a more general setting:

Theorem 8.1. *If H is the closure of an open set in $\mathbf{R}^n$ with a smooth boundary, $n(P)$ denotes the normal vector of of point $P \in \partial H$ and there exists a point O in H such that for all points $P \in \partial H$ $\overrightarrow{OP} \cdot n(P) = d$, then the ratio of the surface-area and volume of H is $\frac{n}{d}$.*

Proof. This time we use Gauss-Ostrogradski. Take the vector-field to be $v(Q) = \overrightarrow{OQ}$. This is not smooth only at point O, so take a ball of radius ε around it and throw it out from set H. What remains we denote by H'. The divergence of this vector field is trivially n, so

$$\int_{\partial H'} \overrightarrow{OQ} dn(Q) = nv(H')$$

($v(H)$ is the volume of H) By assumption on the left side we get $s(H)d - s(B_\varepsilon)\varepsilon$ and $v(S') = v(S) - v(B_\varepsilon)$ (B_ε denotes the ball of radius ε in $\mathbf{R}^n$). So by tending to zero with ε we get the desired result. ☐

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